

Universality for Random Tensors

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Abstract

We prove two universality results for random tensors of arbitrary rank D . We first prove that a random tensor whose entries are N^D independent, identically distributed, complex random variables converges in distribution in the large N limit to the same limit as the distributional limit of a Gaussian tensor model. This generalizes the universality of random matrices to random tensors.

We then prove a second, stronger, universality result. Under the weaker assumption that the joint probability distribution of tensor entries is invariant, assuming that the cumulants of this invariant distribution are uniformly bounded, we prove that in the large N limit the tensor again converges in distribution to the distributional limit of a Gaussian tensor model. We emphasize that the covariance of the large N Gaussian is *not* universal, but depends strongly on the details of the joint distribution.

1 Introduction

There are two main versions of universality in probability theory. The ordinary version is the central limit theorem, stating that the (appropriately rescaled) sum of a large number of independent identically distributed (i.i.d.) random variables follows a normal distribution. The second version, or matrix-case, states that the statistics of invariant quantities of an N by N random matrix is independent of the details of the atomic distribution of the coefficients of the matrix. In the large N limit the random matrix converges in distribution to a Gaussian matrix model. In more familiar terms, the eigenvalues density obeys the Wigner semi-circle law under quite general assumptions [1, 2, 3]. Universality extends to details of the statistics of eigenvalues in the large N limit. The spacing of eigenvalues for instance is determined only by the first four moments of the distribution of the matrix entries [4] and follows Dyson's sine law [5, 6].

In the matrix case the invariant moments are traces of polynomials in the matrix. The limit law can be deduced using a Feynman graph representation. In this approach the problem reduces to finding the so-called $1/N$ expansion for random matrices introduced in [7]. This fixes the correct rescaling of the invariant observables and their limit distribution. The statistics of the eigenvalue density appears as a clever gauge-fixed version of this limit in the particular gauge of *diagonal matrices*. The apparent non-Gaussian character of the Dyson-Wigner law is due to the particular form of the Faddeev-Popov determinant which can be computed exactly in this gauge. The resulting Vandermonde determinant governs the eigenvalues repulsion hence Dyson's sine law. But universality does not *require* gauge-fixing.

Although universality can be established under quite general assumptions, in the matrix case there exist invariant probability laws which are not universal [8]. For example any measure which can be written as the exponential of the trace of a polynomial in the matrix has a planar but not necessarily Gaussian large N limit. A Gaussian matrix can be recovered then via the non commutative central limit theorem. Under very general assumptions random matrices become free in the large N limit (this is again a consequence of the $1/N$ expansion), and the central limit theorem ensures that the (appropriately rescaled) sum of a large number of free matrices converges in distribution to a Gaussian matrix [9, 10, 11].

To summarize there are two ingredients which power both universality and freeness for matrices, namely the invariance and the $1/N$ expansion. Random matrices encode a theory of random two dimensional surfaces and are widely applied in physics for the study of integrable systems, exact critical statistical mechanics,

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quantum gravity in two dimensions and the list goes on. Matrices generalize in higher dimensions to tensors. Introduced in the '90 [12, 13] as tools to study random geometries in dimensions higher than two, random tensor models remained an open problem ever since. Although invariant quantities for tensors are well known, until recently no $1/N$ expansion existed for tensors of rank higher than two and no analytic result on these models could be established. The lack of results on random tensor is exemplified by the Gaussian distribution. One can of course easily write a Gaussian distribution for a random tensor. However its large N behavior, that is identifying the appropriate observables (and their scaling) in the large N limit, has not been established prior to this work.

The situation has drastically changed recently and the necessary ingredients for universality have been found for tensors of higher rank, with the discovery of the $1/N$ expansion [14, 15, 16] for *colored* [17, 18] random tensors. The first consequences for statistical mechanics and quantum gravity have been developed, see [19] for a general review of this thriving subject.

In this paper we derive the universality properties associated to this $1/N$ expansion for a unique complex non symmetric tensor. We establish two universality results. The first one is just the straightforward generalization of the universality of the Gaussian measure to tensors with entries i.i.d. random variables. The second one is more powerful. The natural requirement one should impose on the joint distribution of the tensor entries is not independence, but invariance. We show in this paper that if the joint distribution of the entries is invariant and its cumulants are uniformly bounded then in the large N limit the random tensor converges in distribution to the distributional limit of a Gaussian tensor model. This is in contrast with random matrices, and shows in particular that the Gaussian distribution is a more powerful attractor for higher rank tensors than it is for matrices. However we emphasize that the covariance of the large N Gaussian is *not* universal and the large N limits of random tensors are rather subtle. The Gaussianity allows one only to compute all the large N correlations in terms of the large N covariance, but the latter has a very non trivial dependence on the details of the joint distribution of entries. In particular the perturbed Gaussian measures (presented in appendix A) lead to a multitude of continuum limits [20], thus describing infinitely refined geometries, dominated by spherical topologies [19].

Our results cover tensors of arbitrary rank and lay the foundation for the study of random geometries in arbitrary dimensions using random tensors. This study is relevant for critical statistical mechanics, integrability, quantum gravity and so on in more than two dimensions.

The proofs of our results rely on a representation of the cumulants of the joint distribution of tensor entries by *colored* graphs. This representation is of course inspired by the Feynman graphs representation of perturbed Gaussian measures. However, unlike the former, our representation is completely general and applies to all invariant joint distributions of the entries. The precise link between our graphical representation and Feynman graphs is detailed in the appendix A. Of course, the main challenge is not so much to find an appropriate graphical representation, but to compute the contribution of each graph. This requires on one hand to find the appropriate scaling of various cumulants with N , and on the other hand a detailed combinatorial study of the graphs. If one assumes a *uniform* scaling of the cumulants (i.e. all cumulants at a given order scale with the same power of N , irrespective of the associated graph), the scalings presented in this paper are optimal: tensor distributions which violate them do not admit a large N limit. We comment on these scalings and if they can be relaxed in the non uniform case (i.e. when the scaling of a cumulant is allowed to depend on the details of the associated graph) in appendix B.

One interesting question is to combine our graphical representation with the Connes-Kreimer algebra [21, 22] of the usual Feynman graphs, as the trace invariant cumulants have the structure of an antipode of a graph Hopf algebra. A second important open question not addressed in this paper is to find a clever gauge fixing which would generalize correctly the diagonal condition in the matrix case, and to compute the corresponding Faddeev-Popov determinant. This may require to find better “finite- N truncations” of the theory (i.e. better cutoffs in the quantum field theory language), and an appropriate generalization of the notion of eigenvalues and spectrum for tensors.

The proofs we present below are combinatorial and rely heavily on the colored graph representation we introduce. The plan of the paper is as follows. In section 2 we give the relevant definitions and state our two universality theorems. In section 3 we recall the universality for random matrices and its link with the $1/N$ expansion. We use this opportunity to introduce at length the colored graph representation for this more familiar case. Once familiarized with this representation we present a number of combinatorial results concerning colored graphs in the first part of section 4. We subsequently use this combinatorial input to

prove the two universality results for random tensors in the second part of section 4. Thus the subsections 4.1 and 4.2 are mainly review (except lemmas 4 and 5), while the subsections 4.3, 4.4 and 4.5 are entirely new and contain our main results. In the appendix A we give a detailed presentation of the perturbed Gaussian measures for random tensors, both in perturbations (subsection A.1) for the generic case and at full non perturbative level (subsection A.2) for a particular example. Both these subsections present new results.

This paper falls short in many technical points. We do not give a precise definition of infinite tensors, we do not propose a generalization of the diagonal gauge of random matrices, we do not detail the sub leading corrections in N and so on. All these, and many other, topics need to be thoroughly examined and clarified before obtaining a fully fledged theory of random tensors. Our contribution is the derivation of the generic, universal behavior of random tensors at leading order, which is the prerequisite for all such studies.

2 Notations and Main Theorems

A rank D covariant tensor $T_{n^1 \dots n^D}$ (with $n^1, n^2, \dots, n^D \in \{1, \dots, N\}$) can be seen as a collection of N^D complex numbers supplemented by the requirement of covariance under base change. We consider tensors T with *no symmetry property* under permutation of their indices transforming under the external tensor product of D fundamental representations of $U(N)$. In words, the unitary group acts independently on each index of the tensor. The complex conjugate tensor $\bar{T}_{\bar{n}^1 \dots \bar{n}^D}$ is a rank D contravariant tensor

$$T'_{a^1 \dots a^D} = \sum_{n^1 \dots n^D} U_{a^1 n^1} \dots V_{a^D n^D} T_{n^1 \dots n^D}, \quad \bar{T}'_{\bar{a}^1 \dots \bar{a}^D} = \sum_{\bar{n}^1 \dots \bar{n}^D} \bar{U}_{\bar{a}^D \bar{n}^D} \dots \bar{V}_{\bar{a}^1 \bar{n}^1} \bar{T}_{\bar{n}^1 \dots \bar{n}^D}. \quad (1)$$

where we denoted conventionally the indices of the complex conjugated tensor with a bar. We will sometimes denote the D -uple of integers $n^1 \dots n^D$ by \vec{n} and assume (unless otherwise specified) $D \geq 3$.

Among the invariants one can build out of T and \bar{T} we will deal in the sequel exclusively with **trace invariants**. The trace invariants are built by contracting in all possible ways pairs of covariant and contravariant indices in a product of tensor entries. We write such a trace invariant formally as

$$\text{Tr}(T, \bar{T}) = \sum \prod \delta_{n^1 \bar{n}^1} T_{n^1 \dots} \dots \bar{T}_{\bar{n}^1 \dots}, \quad (2)$$

where all indices are saturated. By the fundamental theorem of classical invariants of $U(N)$, the trace invariants form a basis in the space of invariant polynomials in the tensor entries (see [23] for a direct proof relying on averaging over the unitary group) hence in particular the probability distribution of a random tensor is encoded in their expectations.

Remark that a trace invariant has necessarily the same number of T and \bar{T} and that an index n^i is always contracted with an index \bar{n}^i . A trace invariant can be represented as a bipartite closed D -colored graph (or simply a D -colored graph).

Definition 1. A **bipartite closed D -colored graph** is a graph $\mathcal{B} = (\mathcal{V}(\mathcal{B}), \mathcal{E}(\mathcal{B}))$ with vertex set $\mathcal{V}(\mathcal{B})$ and edge set $\mathcal{E}(\mathcal{B})$ such that:

- $\mathcal{V}(\mathcal{B})$ is bipartite, i.e. there exists a partition of the vertex set $\mathcal{V}(\mathcal{B}) = \mathcal{A}(\mathcal{B}) \cup \bar{\mathcal{A}}(\mathcal{B})$, such that for any element $l \in \mathcal{E}(\mathcal{B})$, then $l = (v, \bar{v})$ with $v \in \mathcal{A}(\mathcal{B})$ and $\bar{v} \in \bar{\mathcal{A}}(\mathcal{B})$. Their cardinalities satisfy $|\mathcal{V}(\mathcal{B})| = 2|\mathcal{A}(\mathcal{B})| = 2|\bar{\mathcal{A}}(\mathcal{B})|$.
- The edge set is partitioned into D subsets $\mathcal{E}(\mathcal{B}) = \bigcup_{i=1}^D \mathcal{E}^i(\mathcal{B})$, where $\mathcal{E}^i(\mathcal{B}) = \{l^i = (v, \bar{v})\}$ is the subset of edges with color i .
- It is D -regular (all vertices are D -valent) with all edges incident to a given vertex having distinct colors.

To draw the graph associated to a trace invariant we represent every $T_{n^1 \dots n^D}$ by a white vertex v and every $\bar{T}_{\bar{n}^1 \dots \bar{n}^D}$ by a black vertex \bar{v} . We promote the positions of an index to a *color*, thus n^1 has color 1, n^2 has color 2 and so on. The contraction of an index n^i on $T_{n^1 \dots n^D}$ with an index \bar{n}^i of $\bar{T}_{\bar{n}^1 \dots \bar{n}^D}$ is represented by a line $l^i = (v, \bar{v}) \in \mathcal{E}^i(\mathcal{B})$ connecting the vertex v (representing $T_{n^1 \dots n^D}$) with the vertex \bar{v} (representing $\bar{T}_{\bar{n}^1 \dots \bar{n}^D}$). The lines inherit the color of the index, i , and always connect a black and a white vertex. Some examples of trace invariants for rank 3 tensors are represented in figure 1. Every trace invariant can be

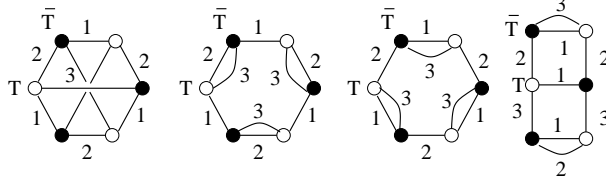


Figure 1: Graphical representation of trace invariants.

written as

$$\text{Tr}_{\mathcal{B}}(T, \bar{T}) = \sum_{n, \bar{n}} \delta_{n\bar{n}}^{\mathcal{B}} \prod_{v, \bar{v} \in \mathcal{V}(\mathcal{B})} T_{\bar{n}_v} \bar{T}_{\bar{n}_{\bar{v}}}, \quad \delta_{n\bar{n}}^{\mathcal{B}} = \prod_{i=1}^D \prod_{l^i=(v, \bar{v}) \in \mathcal{E}^i(\mathcal{B})} \delta_{n_v^i \bar{n}_{\bar{v}}^i}. \quad (3)$$

We call the product $\delta_{n\bar{n}}^{\mathcal{B}}$ encoding the pattern of contraction of the indices the **trace invariant operator** associated to the graph \mathcal{B} [24]. The trace invariant associated to a graph \mathcal{B} factors over its connected components \mathcal{B}_ρ . We call a trace invariant whose associated graph is connected a **connected trace invariant** (or a single trace invariant).

Definition 2. The **faces** of a D -colored graph \mathcal{B} are its connected subgraphs with two colors. We denote $\mathcal{F}^{(i,j)}(\mathcal{B})$ the set of faces with colors i and j of \mathcal{B} , and $F^{ij}(\mathcal{B}) = |\mathcal{F}^{(i,j)}(\mathcal{B})|$ their number. The d -**bubbles** of a graph are its connected subgraphs with d colors.

A colored graph is a cellular complex with cells given by the d -bubbles. In fact it can be shown that it is an abstract simplicial complex, and even more, a simplicial pseudo manifold [17, 18], see also appendix A.

A random tensor is a collection of N^D complex random variables. We consider only even distributions, that is the moments of the joint distribution of tensor entries are non zero only if the numbers of T and \bar{T} insertions are equal. We denote the joint moment of $2k$ tensor entries by $\mu(T_{\bar{n}_1}, \bar{T}_{\bar{n}_1} \dots T_{\bar{n}_k}, \bar{T}_{\bar{n}_k})$. The cumulants of the joint distribution of tensor entries are defined implicitly by

$$\mu(T_{\bar{n}_1}, \bar{T}_{\bar{n}_1} \dots T_{\bar{n}_k}, \bar{T}_{\bar{n}_k}) = \sum_{\pi} \prod_{\alpha=1}^{|\alpha|} \kappa_{2k(\alpha)}[T_{\bar{n}_{\alpha_1}}, \bar{T}_{\bar{n}_{\alpha_1}} \dots], \quad (4)$$

where π runs over the partitions of the set of $2k$ points $\mathcal{V} = \{1 \dots k, \bar{1} \dots \bar{k}\}$ into $|\alpha|$ disjoint bipartite subsets $\mathcal{V}(\alpha) = \{\alpha_1, \dots, \bar{\alpha}_1, \dots\}$ for $\alpha = 1, 2, \dots, |\alpha|$, $|\alpha| \leq k$ with cardinality $|\mathcal{V}(\alpha)| = 2k(\alpha)$. As the partition in a unique set appears only once, the cumulants can be computed in term of the moments. Note that $\sum_{\alpha=1}^{|\alpha|} k(\alpha) = k$.

We will define a trace invariant distribution as a distribution whose *cumulants* are trace invariant operators. We will allow in this definition trace invariant operators which correspond to *disconnected* graphs. At first sight it might seem rather surprising that according to our definition a cumulant (a connected moment) can be expressed as a sum over disconnected graphs. First, the case when the cumulants expand only in connected graphs is certainly a particularization of this more general case. Second, and most importantly, it is in fact natural to allow disconnected graphs into the expansion of a cumulant in invariants. This is clear when dealing with perturbed Gaussian measures in appendix A, both at the perturbative and at the non perturbative level. In perturbations this is seen as follows: moments expand in Feynman graphs, and cumulants (connected moments) expand in connected Feynman graphs \mathcal{G} . However the pattern of contraction of the tensor indices associated to a Feynman graph \mathcal{G} is encoded in its *boundary* graph, $\mathcal{B} = \partial\mathcal{G}$ (a precise definition of the boundary graph is given section 4.2). It turns out that a Feynman graph \mathcal{G} can be connected (thus contributing to a cumulant), and have a disconnected boundary graph $\partial\mathcal{G}$ (as shown figure 6). In order to include the perturbed Gaussian measures one must allow disconnected graphs in the expansion of a cumulant. At the non perturbative level this can be seen as a consequence of the invariance of the cumulants under unitary transformations (see the proof of theorem 8). The same phenomenon appears in the more familiar case of random matrices: at finite N one obtains contributions to the cumulants

corresponding to connected Feynman graphs having two or more external faces (“multi loop observables” in the physics literature). Each external face is a connected component of the boundary graph. However such contributions are penalized in the scaling with N .

We need some more notations. We denote \mathcal{B} a generic D -colored graph with $2k(\mathcal{B})$ vertices labeled $1, \dots, k(\mathcal{B}), \bar{1}, \dots, \bar{k}(\mathcal{B})$. We also denote $\rho(\mathcal{B})$ the number of connected components (labeled \mathcal{B}_ρ) of \mathcal{B} , and $2k(\mathcal{B}_\rho)$ the number of vertices of the connected component \mathcal{B}_ρ . We have $\sum_{\rho=1}^{\rho(\mathcal{B})} k(\mathcal{B}_\rho) = k(\mathcal{B})$ and every graph \mathcal{B} has an associated partition of the vertex set $\{1, \dots, k(\mathcal{B}), \bar{1}, \dots, \bar{k}(\mathcal{B})\}$ into $\rho(\mathcal{B})$ disjoint bipartite subsets of cardinality $2k(\mathcal{B}_\rho)$, $\rho = 1, \dots, \rho(\mathcal{B})$.

Definition 3. *The probability distribution μ of the N^D complex random variables $\mathbb{T}_{\vec{n}}$ is called **trace invariant** if its **cumulants** are linear combinations of trace invariant operators,*

$$\kappa_{2k}[\mathbb{T}_{\vec{n}_1}, \bar{\mathbb{T}}_{\vec{n}_1} \dots \mathbb{T}_{\vec{n}_k}, \bar{\mathbb{T}}_{\vec{n}_k}] = \sum_{\mathcal{B}, k(\mathcal{B})=k} \mathfrak{K}(\mathcal{B}, \mu) \prod_{\rho=1}^{\rho(\mathcal{B})} \delta_{n\bar{n}}^{\mathcal{B}_\rho}, \quad (5)$$

where the sum runs over **all** the D -colored graphs \mathcal{B} with $2k$ vertices.

To compute the joint moments of a trace invariant distribution one has to perform two expansion: first the expansion of the joint moments in cumulants and second the expansion of the cumulants themselves in graphs. We are interested in the large N behavior of a trace invariant probability measure μ_N . In order for such a limit to exist, the cumulants of μ_N must scale with N . There are two main cases. Either the scaling with N is *uniform*, that is it insensitive to all but the roughest features of the graph \mathcal{B} or it depends on the details of \mathcal{B} . We will deal in the main body of this paper with the first case, and briefly discuss the second case in appendix B. We denote

$$\frac{\mathfrak{K}(\mathcal{B}, \mu_N)}{N^{-2(D-1)k(\mathcal{B})+D-\rho(\mathcal{B})}} \equiv K(\mathcal{B}, N) \quad (6)$$

There exists a unique D -colored graph with 2 vertices (all its D lines necessarily connect the two vertices). We call it the D -dipole and denote it $\mathcal{B}^{(2)}$. We call $K(\mathcal{B}^{(2)}, N)$ the covariance of the distribution μ_N .

Definition 4. *We say that the trace invariant probability distribution μ_N is **properly uniformly bounded** at large N if*

$$\lim_{N \rightarrow \infty} K(\mathcal{B}^{(2)}, N) = K(\mathcal{B}^{(2)}) < \infty, \quad K(\mathcal{B}, N) \leq K(\mathcal{B}), \quad \forall \mathcal{B} \neq \mathcal{B}^{(2)} \text{ and } N \text{ large enough}. \quad (7)$$

We will establish our universality results for properly uniformly bounded distributions. A natural question one can ask at this point is if, in particular examples, proper uniform boundedness is easy to establish. This question is addressed in appendix A. We first show that uniform boundedness holds *perturbatively* for all perturbed Gaussian measures. Indeed for such measures the cumulants can be expressed as sums over Feynman graphs and in appendix A.1 we show that each graph respects the proper uniform bound. However this is not yet a proof: in order to establish proper uniform boundedness of a cumulant one must deal with the sum over all Feynman graphs. Sums over graphs are notoriously difficult to control (the perturbative series are not summable, but only Borel summable), and promoting a perturbative bound to a bound at the full non perturbative level is the object of constructive field theory [25]. We will prove in appendix A.2 that the proper uniform bound on the *full resummed cumulants* holds for a measure perturbed by a quartic invariant. The full constructive bounds on cumulants for arbitrary polynomially perturbed Gaussian measures can be achieved by an appropriate generalization of the techniques discussed in appendix A.2. We emphasize that once constructive bounds are established they *always reproduce* the scaling with N of the perturbative bounds, hence the perturbative uniform bounds established in appendix A.1 should hold for the full resummed cumulants also in the general case.

The trace invariance condition of the joint distribution is weaker than the i.i.d. condition. The latter can be seen as supplementing the trace invariant operator $\prod_{\rho=1}^{\rho(\mathcal{B})} \delta_{n\bar{n}}^{\mathcal{B}_\rho}$ by a number of further identifications of indices, imposing that all indices of color i in a cumulant are equal (and modifying appropriately the scaling with N). These extra identifications decrease the number of independent indices and simplify the joint measure.

The normalized Gaussian distribution of covariance σ^2 for a random tensor is the probability measure

$$e^{-N^{D-1} \frac{1}{\sigma^2} \sum_{\vec{n}, \vec{\bar{n}}} \mathbb{T}_{\vec{n}} \delta_{\vec{n}\vec{\bar{n}}} \bar{\mathbb{T}}_{\vec{\bar{n}}} \prod_{\vec{n}} \left(\frac{N^{D-1}}{\sigma^2} \frac{d\mathbb{T}_{\vec{n}} d\bar{\mathbb{T}}_{\vec{\bar{n}}}}{2\pi i} \right)}. \quad (8)$$

It is characterized by the expectations of the connected (single) trace invariants

$$\left\langle \text{Tr}_{\mathcal{B}}(\mathbb{T}, \bar{\mathbb{T}}) \right\rangle_{\sigma^2} = \int \left(\prod_{\vec{n}} \frac{N^{D-1}}{\sigma^2} \frac{d\mathbb{T}_{\vec{n}} d\bar{\mathbb{T}}_{\vec{\bar{n}}}}{2\pi i} \right) e^{-N^{D-1} \frac{1}{\sigma^2} \sum_{\vec{n}, \vec{\bar{n}}} \mathbb{T}_{\vec{n}} \delta_{\vec{n}\vec{\bar{n}}} \bar{\mathbb{T}}_{\vec{\bar{n}}}} \text{Tr}_{\mathcal{B}}(\mathbb{T}, \bar{\mathbb{T}}). \quad (9)$$

It is in fact a non trivial problem to compute the moments of the Gaussian distribution, and we defer it to section 4.3. For now we just mention that for any graph \mathcal{B} with $2k(\mathcal{B})$ vertices there exist two non negative integers, $\Omega(\mathcal{B})$ and $R(\mathcal{B})$ such that

$$\lim_{N \rightarrow \infty} N^{-1+\Omega(\mathcal{B})} \left\langle \text{Tr}_{\mathcal{B}}(\mathbb{T}, \bar{\mathbb{T}}) \right\rangle_{\sigma^2} = \sigma^{2k(\mathcal{B})} R(\mathcal{B}). \quad (10)$$

We call $\Omega(\mathcal{B})$ the convergence order of the invariant \mathcal{B} . The normalization in eq. (8) is the **only normalization** which ensures that the convergence order is positive and, more importantly, **for all** \mathcal{B} , there exists an **infinite** family of invariants (graphs \mathcal{B}') such that $\Omega(\mathcal{B}) = \Omega(\mathcal{B}')$, see lemma 7.

Definition 5. A random tensor \mathbb{T} distributed with the probability measure μ_N **converges in distribution** to the distributional limit of a Gaussian tensor model of covariance σ^2 if the large N limit of the expectation of any connected trace invariant equals the large N Gaussian expectation of the invariant

$$\lim_{N \rightarrow \infty} N^{-1+\Omega(\mathcal{B})} \mu_N \left[\text{Tr}_{\mathcal{B}}(\mathbb{T}, \bar{\mathbb{T}}) \right] = \sigma^{2k(\mathcal{B})} R(\mathcal{B}). \quad (11)$$

This paper establishes two theorems. The first one simply generalizes the universality of random matrices to random tensors:

Theorem 1 (Universality 1). *Let N^D i.i.d. random variables $T_{\vec{n}}$, each of covariance σ^2 . Then, in the large N limit, the tensor $\mathbb{T}_{\vec{n}} = \frac{1}{N^{\frac{D-1}{2}}} T_{\vec{n}}$ converges in distribution to a Gaussian tensor of covariance σ^2 .*

The second universality theorem is:

Theorem 2 (Main Theorem: Universality 2). *Let N^D random variables $\mathbb{T}_{\vec{n}}$ whose joint distribution is trace invariant and properly uniformly bounded of covariance $\mathfrak{K}(\mathcal{B}^{(2)}, \mu_N)$. Then in the large N limit the tensor $\mathbb{T}_{\vec{n}}$ converges in distribution to a Gaussian tensor of covariance $K(\mathcal{B}^{(2)}) = \lim_{N \rightarrow \infty} \mathfrak{K}(\mathcal{B}^{(2)}, \mu_N)$.*

Universality is thus much stronger for random tensors than it is for random matrices. For the latter universality can be established if, for instance, the distribution μ_N is i.i.d, but one achieves various non Gaussian large N limits [8] for trace invariant measures. The limit eigenvalue distributions can be evaluated and it is different from the usual semicircle law (multi cut solutions and so on). A set of matrices whose joint distribution is trace invariant become free in the large N limit. Random tensors exhibit a more powerful universality property: properly uniformly bounded trace invariant distributions become Gaussian in the large N limit. However note that the large N covariance $K(\mathcal{B}^{(2)})$ strongly depends on the details of the joint distribution at finite N . For the case of perturbed Gaussian measures the large N covariance is a sum over an infinite family of Feynman graphs and exhibits various multicritical behaviors [20].

Before proceeding we fix some notations. From now on \mathcal{B} will always designate the invariant whose expectation we evaluate. As we deal only with connected (single trace) invariants, \mathcal{B} will always be a *connected* D colored graph. The graphs $\mathcal{B}(\alpha)$ arise from the expansion of cumulants into trace invariant operators. They are *not* connected. Their connected components are labeled $\mathcal{B}_\rho(\alpha)$.

When evaluating expectations of observables we will introduce $D + 1$ colored graphs (definition 1 with D replaced by $D + 1$). We will call the new color 0. We will use \mathcal{G} as a dustbin notation for *connected* $D + 1$ colored graphs. The lines of the new color 0, denoted $l^0 \in \mathcal{E}^0(\mathcal{G})$, play a special role and will be represented as dashed lines.

3 Random Matrices

We will first detail the case of random matrices. This serves both as motivation and as an opportunity to introduce the appropriate tools for the study of random tensors.

All connected bi-colored graphs with $2k$ vertices are cycles with alternating colors (which we denote \mathcal{B}). The associated trace invariants write

$$\begin{aligned}\delta_{n\bar{n}}^{\mathcal{B}} &= \prod_{i=1}^2 \prod_{l^i=(v,\bar{v}) \in \mathcal{E}^i(\mathcal{B})} \delta_{n_v^i \bar{n}_{\bar{v}}^i} \\ \text{Tr}_{\mathcal{B}}(A, \bar{A}) &= \sum_{n, \bar{n}} \delta_{n\bar{n}}^{\mathcal{B}} \prod_{v, \bar{v} \in \mathcal{V}(\mathcal{B})} A_{\bar{n}_v} \bar{A}_{\bar{n}_{\bar{v}}} \equiv \text{Tr}[(AA^\dagger)^k],\end{aligned}\quad (12)$$

Any invariant function of a generic (i.e. not necessarily hermitian) matrix can be evaluated starting from these trace invariants, as they fix the spectral measure of AA^\dagger .

Gaussian distribution of a random matrix. The Gaussian distribution of a non hermitian random $N \times N$ matrix \mathbb{A} of covariance 1 is the probability measure

$$e^{-N \sum \mathbb{A}_{n^1 n^2} \delta_{n^1 \bar{n}^1} \delta_{n^2 \bar{n}^2} \bar{\mathbb{A}}_{\bar{n}^1 \bar{n}^2}} \prod_{(n^1, n^2)} \left(N \frac{d\mathbb{A}_{n^1 n^2} d\bar{\mathbb{A}}_{\bar{n}^1 \bar{n}^2}}{2\pi i} \right), \quad (13)$$

where the product is taken over all the (complex) entries $\mathbb{A}_{n^1 n^2}$. Note that the exponent can alternatively be written in the more familiar form $N \text{Tr}(\mathbb{A} \mathbb{A}^\dagger)$. The Gaussian distribution is characterized by its expectations in the large N limit,

$$\lim_{N \rightarrow \infty} N^{-1} \left\langle \text{Tr}[(\mathbb{A}^\dagger \mathbb{A})^k] \right\rangle = \frac{1}{k+1} \binom{2k}{k}, \quad (14)$$

It is instructive to prove this. We represent the trace invariant as a colored cycle \mathcal{B} with $2k$ vertices

$$\left\langle \text{Tr}[(\mathbb{A}^\dagger \mathbb{A})^k] \right\rangle = \sum_{n, \bar{n}} \delta_{n\bar{n}}^{\mathcal{B}} \left\langle \prod_{v, \bar{v} \in \mathcal{V}(\mathcal{B})} \mathbb{A}_{\bar{n}_v} \bar{\mathbb{A}}_{\bar{n}_{\bar{v}}} \right\rangle. \quad (15)$$

The Gaussian expectation of a product of matrix entries is a sum over pairings (Wick contractions in the physics language) of products of covariances. If two matrix entries are paired by a covariance we connect them by a dashed line (to which we associate by convention the color 0). A pairing is then represented as a (Feynman) graph \mathcal{G} .

Definition 6. We call a graph with 3 colors \mathcal{G} a **covering graph** of \mathcal{B} if \mathcal{G} reduces to \mathcal{B} by deleting the lines of color 0, $\mathcal{G} \setminus \mathcal{E}^0(\mathcal{G}) = \mathcal{B}$.

The contraction of two entries $\mathbb{A}_{n^1 n^2}$ and $\bar{\mathbb{A}}_{\bar{n}^1 \bar{n}^2}$ with the Gaussian measure (13) comes to replacing them by $\frac{1}{N} \delta_{n^1 \bar{n}^1} \delta_{n^2 \bar{n}^2}$, hence each line of color 0, $l^0 = (v, \bar{v}) \in \mathcal{E}^0(\mathcal{G})$, will bring a factor $\frac{1}{N} \delta_{n_v^1 \bar{n}_{\bar{v}}^1} \delta_{n_v^2 \bar{n}_{\bar{v}}^2}$.

The graph of the invariant \mathcal{B} has two colors 1 and 2, while a covering graph \mathcal{G} has three colors: 1, 2 and the extra color 0 of the dashed lines. An example of a covering graph \mathcal{G} contributing to the expectation of $\text{Tr}[(\mathbb{A}^\dagger \mathbb{A})^3]$ is presented in figure 2.

The expectation of \mathcal{B} becomes a sum over all covering graphs \mathcal{G}

$$\left\langle \text{Tr}[(\mathbb{A}^\dagger \mathbb{A})^k] \right\rangle = \sum_{n, \bar{n}} \left(\prod_{i=1}^2 \prod_{l^i=(v,\bar{v}) \in \mathcal{E}^i(\mathcal{B})} \delta_{n_v^i \bar{n}_{\bar{v}}^i} \right) \sum_{\mathcal{G}, \mathcal{G} \setminus \mathcal{E}^0(\mathcal{G}) = \mathcal{B}} \prod_{l^0=(v,\bar{v}) \in \mathcal{E}^0(\mathcal{G})} \frac{1}{N} \delta_{n_v^1 \bar{n}_{\bar{v}}^1} \delta_{n_v^2 \bar{n}_{\bar{v}}^2}, \quad (16)$$

and, as the lines of color 1 and 2 of any such \mathcal{G} are in fact the lines of color 1 and 2 of \mathcal{B}

$$\left\langle \text{Tr}[(\mathbb{A}^\dagger \mathbb{A})^k] \right\rangle = \sum_{\mathcal{G}, \mathcal{G} \setminus \mathcal{E}^0(\mathcal{G}) = \mathcal{B}} \sum_{n, \bar{n}} \left(\prod_{i=1}^2 \prod_{l^i=(v,\bar{v}) \in \mathcal{E}^i(\mathcal{G})} \delta_{n_v^i \bar{n}_{\bar{v}}^i} \right) \left(\prod_{l^0=(v,\bar{v}) \in \mathcal{E}^0(\mathcal{G})} \frac{1}{N} \delta_{n_v^1 \bar{n}_{\bar{v}}^1} \delta_{n_v^2 \bar{n}_{\bar{v}}^2} \right). \quad (17)$$

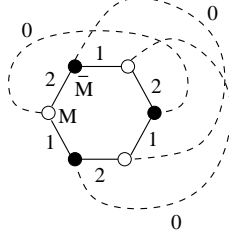


Figure 2: A covering graph \mathcal{G} of an observable \mathcal{B} .

To evaluate the contribution of a graph \mathcal{G} one must evaluate the number of independent sums over the matrix indices n, \bar{n} . The Kronecker δ compose along the faces (bi-colored circuits) of colors 01 and 02 and yield an independent free sum for each such face. As we have exactly k lines of color 0 we get

$$\left\langle \text{Tr}[(\mathbb{A}^\dagger \mathbb{A})^k] \right\rangle = \sum_{\mathcal{G}, \mathcal{G} \setminus \mathcal{E}^0(\mathcal{G}) = \mathcal{B}} \frac{1}{N^k} N^{F^{01}(\mathcal{G}) + F^{02}(\mathcal{G})}. \quad (18)$$

Note that the face 12 corresponding to the circuit \mathcal{B} with colors 12 (hence to the observable itself) does not bring any sum. The graph \mathcal{G} has $2k$ vertices (k black and k white), $3k$ lines (k dashed lines of color 0 and k solid lines for each of the colors 1 and 2) and faces $(F^{01}(\mathcal{G}) + F^{02}(\mathcal{G}))$ representing free sums and $F^{12}(\mathcal{G}) = 1$ with no sum). The Euler character of \mathcal{G} is

$$2k - 3k + F^{01}(\mathcal{G}) + F^{02}(\mathcal{G}) + 1 = 2 - 2g(\mathcal{G}) \Rightarrow -1 - k + F^{01}(\mathcal{G}) + F^{02}(\mathcal{G}) = -2g(\mathcal{G}). \quad (19)$$

It follows that in the large N limit only graphs \mathcal{G} of genus $g(\mathcal{G}) = 0$ contribute. We call such graphs **minimal covering graphs** of \mathcal{B} . Equivalently they can be seen as the covering graphs of \mathcal{B} with maximal number of faces $F^{01}(\mathcal{G}) + F^{02}(\mathcal{G})$. Thus

$$\lim_{N \rightarrow \infty} N^{-1} \left\langle \text{Tr}(\mathbb{A}^\dagger \mathbb{A})^k \right\rangle = R_k, \quad (20)$$

where R_k counts the number of minimal (planar) covering graphs \mathcal{G} , $\mathcal{G} \setminus \mathcal{E}^0(\mathcal{G}) = \mathcal{B}$. It is easy to see that $R_1 = 1$ and $R_{k+1} = \sum_{p=0}^k R_p R_{k-p}$, thus $R_k = \frac{1}{k+1} \binom{2k}{k}$, i.e. R_k are the Catalan numbers. The normalization of the Gaussian is canonical, and not a matter of choice: any other normalization leads either to infinite or to zero expectations in the large N limit.

3.1 Universality for Random Matrices

In order to introduce the ideas we will use later to prove the universality properties of random higher rank tensors we present below the classical universality of random matrices using this graphical representation.

Theorem 3. *Let M be a matrix with entries i.i.d. complex random variables with centered distributions of unit covariance. In the large N limit, the matrix $\mathbb{M} = \frac{1}{\sqrt{N}} M$ converges in distribution to a random matrix distributed on a Gaussian.*

Proof: Non hermitian matrices whose entries are i.i.d. complex random variables are called random covariance matrices [8]. The moments of the matrix \mathbb{M} write

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \mu_N \left[\text{Tr}(\mathbb{M} \mathbb{M}^\dagger)^k \right] &= \lim_{N \rightarrow \infty} \frac{1}{N^{1+k}} \mu_N \left[\text{Tr}(M M^\dagger)^k \right] \\ &= \lim_{N \rightarrow \infty} \frac{1}{N^{1+k}} \sum_{n, \bar{n}} \delta_{n\bar{n}}^{\mathcal{B}} \mu_N \left[\prod_{v, \bar{v} \in \mathcal{V}(\mathcal{B})} M_{\vec{n}_v} \bar{M}_{\vec{n}_{\bar{v}}} \right] \\ &= \lim_{N \rightarrow \infty} \frac{1}{N^{1+k}} \sum_{n, \bar{n}} \delta_{n\bar{n}}^{\mathcal{B}} \sum_{\pi} \kappa_{\pi} \left[\prod_{v, \bar{v} \in \mathcal{V}(\mathcal{B})} M_{\vec{n}_v} \bar{M}_{\vec{n}_{\bar{v}}} \right], \end{aligned} \quad (21)$$

where we denoted κ_π the product of cumulants associated to the partition π . As the entries are independent, the only non zero cumulants are $\kappa_{2q}[(M_{ij})^q(\bar{M}_{ij})^q]$. Like in the Gaussian case, each cumulant will introduce constraints on the number of independent sums. We slightly extend our graphical representation. If two matrix entries are connected by a two point cumulant we connect them, as in the Gaussian case, by a dashed line of color 0. If four (or more) matrix entries are connected by a cumulant, all the four (or more) matrix elements have the same indices. We will employ a simple trick to represent such cumulants, namely we will connect the matrix entries two by two (a M and a \bar{M}) by dashed lines of color 0 and keep in mind that the indices are further identified. The pairing is not canonical, and in order to control the sub leading contributions one needs to improve this graphical representation and track carefully the higher order cumulants. However at leading order we just need a rough estimate of the number of independent sums in an observable and a non canonical pairing suffices.

The graphs \mathcal{G} we obtain are covering graphs of \mathcal{B} , $\mathcal{G} \setminus \mathcal{E}^0(\mathcal{G}) = \mathcal{B}$. We have (at most) an independent sum over an index corresponding to the faces 01 and 02 (potentially less if several dashed lines correspond to a higher order cumulant). In the large N limit only planar graphs (minimal covering graphs) contribute. Furthermore, if such a planar graph corresponds to a factorization with a fourth (or higher) order cumulants, some of the faces 01 and 02 are further identified (as a pair of distinct lines of color 0 on a planar graph with a unique face 12 can never share both faces 01 and 02), hence the number of independent sums is strictly smaller than $F^{01}(\mathcal{G}) + F^{02}(\mathcal{G})$ in this case. It follows that the only surviving contributions in the large N correspond to planar graphs in which all dashed lines come from a second order cumulant

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mu \left[\text{Tr}(\mathbb{M} \mathbb{M}^\dagger)^k \right] = \lim_{N \rightarrow \infty} \sum_{\mathcal{G}, \mathcal{G} \setminus \mathcal{E}^0(\mathcal{G}) = \mathcal{B}} \left(\kappa_2 [M_{ij} \bar{M}_{ij}] \right)^k = R_k, \quad (22)$$

where we used the fact that the covariance of the atomic distribution is one. □

In the case of matrices we have another clever set of observables, the eigenvalues of the matrix $\mathbb{M} \mathbb{M}^\dagger$, which are non-polynomial functions of the generators. Passing to this set of variables is analog to writing the theory in a particular gauge and the corresponding Faddeev-Popov determinant results from the integration over the unitary group with the Haar measure. The result is the well known Vandermonde polynomial.

We now relax the requirement of independence and require only trace invariance of the joint distribution of the entries. Thus in eq. (21)

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mu \left[\text{Tr}(\mathbb{M} \mathbb{M}^\dagger)^k \right] = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n, \bar{n}} \delta_{n\bar{n}}^{\mathcal{B}} \sum_{\pi} \kappa_{\pi} \left[\prod_{v, \bar{v} \in \mathcal{V}(\mathcal{B})} \mathbb{M}_{\bar{n}_v} \bar{\mathbb{M}}_{\bar{n}_{\bar{v}}} \right], \quad (23)$$

one substitutes for each set in the partition π the properly uniformly bounded trace invariant cumulants of eq. (5) and (7)

$$\kappa_{2k(\alpha)} [\mathbb{M}_{\bar{n}_1}, \bar{\mathbb{M}}_{\bar{n}_1} \dots \bar{\mathbb{M}}_{\bar{n}_{k(\alpha)}}] = \sum_{\mathcal{B}(\alpha), k(\mathcal{B}(\alpha))=k(\alpha)} N^{-2k(\mathcal{B}(\alpha))+2-\rho(\mathcal{B}(\alpha))} K(\mathcal{B}(\alpha), N) \prod_{\rho=1}^{\rho(\mathcal{B}(\alpha))} \delta_{n\bar{n}}^{\mathcal{B}_\rho(\alpha)}. \quad (24)$$

The index $\alpha = 1, \dots, |\alpha|$ tracks the cumulant $\kappa_{2k(\alpha)}$ appearing in the expansion of the joint moment. The index $\rho = 1, \dots, \rho(\mathcal{B}(\alpha))$ labels (at fixed $\mathcal{B}(\alpha)$) the connected components $\mathcal{B}_\rho(\alpha)$ in the expansion of $\kappa_{2k(\alpha)}$ in trace invariants.

When evaluating the expectation of a trace observables, the sum over partitions π becomes a sum over graphs \mathcal{G} . The graph \mathcal{G} representing a term in the sum is constructed as follows. First one draws the observable \mathcal{B} and an invariant $\mathcal{B}(\alpha)$ (with connected components $\mathcal{B}_\rho(\alpha)$) for each $\kappa_{2k(\alpha)}$ for $\alpha = 1, \dots, |\alpha|$.

Note that $\sum_{\rho=1}^{\rho(\mathcal{B}(\alpha))} k(\mathcal{B}_\rho(\alpha)) = k(\alpha)$ and $\sum_{\alpha=1}^{|\alpha|} k(\alpha) = k$. As a matter of convention we flip all the black and white vertices of \mathcal{B} . Note that in this graphical representation all the original vertices of \mathcal{B} are doubled: every vertex appears once in \mathcal{B} and once in some $\mathcal{B}_\rho(\alpha)$. We connect every vertex representing a matrix entry \mathbb{M} in \mathcal{B} with the vertex representing the same matrix entry \mathbb{M} in the corresponding $\mathcal{B}_\rho(\alpha)$ by a fictitious

dashed line of color 0. We explicit the label α of the D -colored graphs $\mathcal{B}_\rho(\alpha)$. Some example are presented in figure 3.

We thus construct a closed connected graph \mathcal{G} having three colors, 0, 1 and 2. As we flipped the black and white vertices on \mathcal{B} , all lines of color 0 in \mathcal{G} will connect a black and a white vertex. We call a graph built in this way a **doubled graph**. The sums over partitions π and invariants $\mathcal{B}(\alpha)$ in equations (23) and (24) becomes a sum over all doubled graphs \mathcal{G} one can build starting from \mathcal{B} which we denote $\mathcal{G} \supset \mathcal{B}$. Starting from a given \mathcal{G} one readily identifies $\mathcal{B}, \mathcal{B}_\rho(\alpha)$ and $\rho(\mathcal{B}(\alpha))$: the observable \mathcal{B} is the subgraph with colors $1, \dots, D$ of \mathcal{G} having no label α , all the other subgraphs with colors $1, \dots, D$ of \mathcal{G} represent the various $\mathcal{B}_\rho(\alpha)$'s, that is $\mathcal{G} \setminus \mathcal{E}^0(\mathcal{G}) = \mathcal{B} \cup_{\alpha=1}^{|\alpha|} \left(\bigcup_{\rho=1}^{\rho(\mathcal{B}(\alpha))} \mathcal{B}_\rho(\alpha) \right)$, and $\rho(\mathcal{B}(\alpha))$ is the number of connected components of \mathcal{G} sharing the same label α .

This graphical representation applies to all trace invariant measures. We will see in appendix A the precise relation between the usual Feynman graphs for perturbed Gaussian measures and these doubled graphs, but we warn the reader that this relation is more subtle than it might appear at first sight.

Some doubled graphs contributing to the observable $\text{Tr}[(\mathbb{M}\mathbb{M}^\dagger)^3]$ are given in figure 3. The face 12 associated to \mathcal{B} is the one with six vertices, while the faces 12 with four and two vertices correspond to various $\mathcal{B}_\rho(\alpha)$. We have also identified on the drawings the various cumulants α to which each connected component $\mathcal{B}_\rho(\alpha)$ belongs. Thus on the left hand side of figure 3 we represented a contribution from two cumulants. The first one is a two point cumulant $k(\mathcal{B}(1)) = 1$, and the second one is a four point cumulant $k(\mathcal{B}(2)) = 2$. The invariant for the first cumulant has a connected component $\rho(\mathcal{B}(1)) = 1$ with two vertices $k(\mathcal{B}_1(1)) = 1$. The invariant for the second cumulant has also one connected component $\rho(\mathcal{B}(2)) = 1$ but this time with four vertices $k(\mathcal{B}_1(2)) = 2$. On the right of figure 3 we presented a contribution coming from *the same* two cumulants, $k(\mathcal{B}(1)) = 1$, $k(\mathcal{B}(2)) = 2$. The invariant for the first cumulant has again a connected component $\rho(\mathcal{B}(1)) = 1$ with two vertices $k(\mathcal{B}_1(1)) = 1$. But this time the invariant for the second cumulant has two connected components $\rho(\mathcal{B}(2)) = 2$, each with two vertices $k(\mathcal{B}_1(2)) = 1, k(\mathcal{B}_2(2)) = 1$.

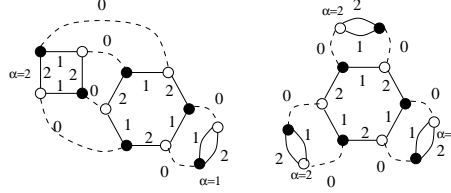


Figure 3: Doubled graphs contributing to an observable.

To evaluate the contribution of a graph \mathcal{G} to the expectation of an observable one must remember that we first divide the $2k$ points among $|\alpha|$ cumulants, and subsequently the $2k(\alpha)$ points in every cumulant are subdivided into $\rho(\mathcal{B}(\alpha))$ connected graphs $\mathcal{B}_\rho(\alpha)$. As the lines of color 0 connect two copies of the same vertex, the indices of their end points are identical, hence each $l^0 = (v, \bar{v}) \in \mathcal{E}^0(\mathcal{G})$ contributes $\delta_{n_v^1 \bar{n}_{\bar{v}}^1} \delta_{n_v^2 \bar{n}_{\bar{v}}^2}$.

The expectation of an invariant observable becomes

$$\begin{aligned}
& \frac{1}{N} \mu \left(\text{Tr}(\mathbb{M}^\dagger \mathbb{M})^k \right) = \\
& \frac{1}{N} \sum_{\mathcal{G} \supset \mathcal{B}, \mathcal{G} \setminus \mathcal{E}^0(\mathcal{G}) = \mathcal{B} \cup_{\alpha=1}^{|\alpha|} \left(\bigcup_{\rho=1}^{\rho(\mathcal{B}(\alpha))} \mathcal{B}_\rho(\alpha) \right)} N^{\sum_{\alpha=1}^{|\alpha|} \left(-2k(\mathcal{B}(\alpha)) + 2 - \rho(\mathcal{B}(\alpha)) \right)} \prod_{\alpha=1}^{|\alpha|} K(\mathcal{B}(\alpha), N) \\
& \times \sum_{n, \bar{n}} \left(\delta_{n\bar{n}}^{\mathcal{B}} \prod_{\alpha=1}^{|\alpha|} \prod_{\rho=1}^{\rho(\mathcal{B}(\alpha))} \delta_{n\bar{n}}^{\mathcal{B}_\rho(\alpha)} \right) \prod_{l^0 = (v, \bar{v}) \in \mathcal{E}^0(\mathcal{G})} \delta_{n_v^1 \bar{n}_{\bar{v}}^1} \delta_{n_v^2 \bar{n}_{\bar{v}}^2} .
\end{aligned} \tag{25}$$

The total operator $\left(\delta_{n\bar{n}}^{\mathcal{B}} \prod_{\alpha=1}^{|\alpha|} \prod_{\rho=1}^{\rho(\mathcal{B}(\alpha))} \delta_{n\bar{n}}^{\mathcal{B}_\rho(\alpha)} \right)$ explains our representation in doubled graphs: one must keep track of the observable \mathcal{B} , the cumulants $\kappa_{2k(\alpha)}$ and the graphs $\mathcal{B}_\rho(\alpha)$ in order to compute the contribu-

tion of a term to the expectation of the observable. In particular this requires the doubling of the vertices. Substituting the trace invariant operators $\delta_{n\bar{n}}^{\mathcal{B}}$ and $\delta_{n\bar{n}}^{\mathcal{B}_\rho(\alpha)}$ eq. (25) becomes

$$\frac{1}{N} \sum_{\mathcal{G} \supset \mathcal{B}, \mathcal{G} \setminus \mathcal{E}^0(\mathcal{G}) = \mathcal{B} \cup_{\alpha=1}^{|\alpha|} \left(\cup_{\rho=1}^{\rho(\mathcal{B}(\alpha))} \mathcal{B}_\rho(\alpha) \right)} N^{\sum_{\alpha=1}^{|\alpha|} \left(-2k(\mathcal{B}(\alpha)) + 2 - \rho(\mathcal{B}(\alpha)) \right)} \prod_{\alpha=1}^{|\alpha|} K(\mathcal{B}(\alpha), N) \sum_{n, \bar{n}} \left(\prod_{i=1}^2 \prod_{l^i = (v, \bar{v}) \in \mathcal{E}^i \left(\mathcal{B} \cup_{\alpha=1}^{|\alpha|} \left(\cup_{\rho=1}^{\rho(\mathcal{B}(\alpha))} \mathcal{B}_\rho(\alpha) \right) \right)} \delta_{n_v^i \bar{n}_{\bar{v}}^i} \right) \prod_{l^0 = (v, \bar{v}) \in \mathcal{E}^0(\mathcal{G})} \delta_{n_v^1 \bar{n}_{\bar{v}}^1} \delta_{n_v^2 \bar{n}_{\bar{v}}^2}, \quad (26)$$

and noting again that the lines of colors 1 and 2 of $\mathcal{B} \cup_{\alpha=1}^{|\alpha|} \left(\cup_{\rho=1}^{\rho(\mathcal{B}(\alpha))} \mathcal{B}_\rho(\alpha) \right)$ are exactly the lines of color 1 and 2 in \mathcal{G} , we see again that the Kronecker δ 's compose along the faces of colors 01 and 02 of \mathcal{G} , thus

$$\frac{1}{N} \mu \left(\text{Tr}(\mathbb{M}^\dagger \mathbb{M})^k \right) = \sum_{\mathcal{G}, \mathcal{G} \supset \mathcal{B}} N^{-1-2k + \sum_{\alpha=1}^{|\alpha|} \rho(\mathcal{B}(\alpha)) + F^{01}(\mathcal{G}) + F^{02}(\mathcal{G}) - 2 \sum_{\alpha=1}^{|\alpha|} (\rho(\mathcal{B}(\alpha)) - 1)} \prod_{\alpha=1}^{|\alpha|} K(\mathcal{B}(\alpha), N). \quad (27)$$

The doubled graph \mathcal{G} has $4k$ vertices, $2k$ coming from \mathcal{B} and $2k$ coming from all the $\mathcal{B}_\rho(\alpha)$. It has $1 + \sum_{\alpha=1}^{|\alpha|} \rho(\mathcal{B}(\alpha))$ faces 12, one associated to the observable \mathcal{B} , and one for each $\mathcal{B}_\rho(\alpha)$. Furthermore it has $2k$ lines of color 0, $2k$ lines of color 1 and $2k$ lines of color 2. The Euler character of \mathcal{G} is

$$4k - 6k + 1 + \sum_{\alpha=1}^{|\alpha|} \rho(\mathcal{B}(\alpha)) + F^{01}(\mathcal{G}) + F^{02}(\mathcal{G}) = 2 - 2g(\mathcal{G}), \quad (28)$$

hence the global scaling with N of a term is $N^{-2g(\mathcal{G}) - 2 \sum_{\alpha=1}^{|\alpha|} (\rho(\mathcal{B}(\alpha)) - 1)}$. It follows that \mathcal{G} contributes to expectation of an observable in the large N limit if it is planar and each cumulant $\kappa_{2k(\alpha)}$ contributes exactly one connected invariant $\rho(\mathcal{B}(\alpha)) = 1$. The second condition is easy to understand for perturbed Gaussian measures. As previously stated the disconnected invariants $\mathcal{B}(\alpha)$ correspond to Feynman graphs having more than one external face. Reconnecting the external lines on such a cumulant on the observable \mathcal{B} leads to non planar Feynman graphs, in spite of the fact that the associated doubled graph (which only sees the boundary of the Feynman graph contributing to the cumulant) is planar. This emphasizes the non trivial relation between Feynman graphs and doubled graphs.

The planar graphs contributing to the large N limit possess cumulants of orders between 2 and $2k$ (each cumulant contributing only when its associated invariant is connected), hence the large N distribution of \mathbb{M} is not Gaussian. The restriction of trace invariant measures for matrices to planar graphs has a different effect: one can easily show that matrices distributed according to such measures become free in the large N limit. This is particularly transparent in the combinatorial formulation of free probability theory of [26, 27]. In the large N limit only the free cumulants (defined by restricting the sum in eq. (4) to non crossing partitions) survive, and one can show that (in the large N limit) the mixed free cumulants of a collection of matrices cancel. As one only deals with the $N \rightarrow \infty$ limit, the free cumulants are automatically associated to connected invariants. One example of a random matrix model whose measure is not trace invariant is the Grosse Wulkenhaar model [28] which is only almost trace invariant.

4 Random Tensors

We now go to the core of our paper and the proofs of the two theorems. We start by an account of properties of D and $D + 1$ colored graphs we will use in the sequel. Most of the lemmas we present in subsections 4.1 and 4.2 can be found in [16, 19, 24, 29]. The rest of this section is new.

4.1 $D + 1$ -colored Graphs

The connected (single trace) observables of tensor models are represented by connected D -colored graphs \mathcal{B} . Their expectations are evaluated in terms of $D + 1$ -colored graphs \mathcal{G} , having an extra color 0. We will use the shorthand notation $\hat{0} \equiv \{1, \dots, D\}$.

Consider a *connected* $D + 1$ colored graph \mathcal{G} . To simplify notations we will drop in this subsection as much as possible \mathcal{G} from our notations. Thus the sets of vertices, edges and faces of colors ij (definition 2) of \mathcal{G} are denoted \mathcal{V} , \mathcal{E} and $\mathcal{F}^{(i,j)}$. Furthermore we denote $\mathcal{F} = \cup_{i < j} \mathcal{F}^{(i,j)}$ and $F = |\mathcal{F}|$. We define the jackets [15, 16, 19] of the $D + 1$ -colored graph \mathcal{G} .

Definition 7. A *colored jacket* \mathcal{J} is a 2-subcomplex of \mathcal{G} , labeled by a $(D + 1)$ -cycle τ , such that:

- \mathcal{J} and \mathcal{G} have identical vertex sets, $\mathcal{V}(\mathcal{J}) = \mathcal{V}$;
- \mathcal{J} and \mathcal{G} have identical edge sets, $\mathcal{E}(\mathcal{J}) = \mathcal{E}$;
- the face set of \mathcal{J} is a subset of the face set of \mathcal{G} : $\mathcal{F}(\mathcal{J}) = \cup_{q=0}^D \mathcal{F}^{(\tau^q(0), \tau^{q+1}(0))}$.

For example the jacket associated to the cycle $(0, 1, 2 \dots D)$ contains the faces $(01)(12)(23) \dots (D0)$. It is evident that \mathcal{J} and \mathcal{G} have the same connectivity. A given jacket is independent of the overall orientation of the cycle, meaning that the jackets are in one-to-two correspondence with $(D + 1)$ -cycles. Therefore, the number of independent jackets is $D!/2$ and the number of jackets containing a given face is $(D - 1)!$.¹

The jacket has the structure of a *ribbon graph*, [30], as each edge of \mathcal{J} lies on the boundary of two of its faces. A ribbon line that separates the two faces, $(\tau^{-1}(i), i)$ and $(i, \tau(i))$ inherits the color i of the line in \mathcal{G} . Ribbon graphs are well-known to correspond to Riemann surfaces [30], and so the same holds for jackets. Given this, we can compute the Euler character of the jacket, $\chi(\mathcal{J}) = |\mathcal{F}(\mathcal{J})| - |\mathcal{E}(\mathcal{J})| + |\mathcal{V}(\mathcal{J})| = 2 - 2g(\mathcal{J})$, where $g(\mathcal{J})$ is the genus of the jacket.²

Definition 8. The **convergence degree** (or simply **degree**) of a graph \mathcal{G} is $\omega(\mathcal{G}) = \sum_{\mathcal{J}} g(\mathcal{J})$, where the sum runs over all the $D!/2$ distinct jackets \mathcal{J} of \mathcal{G} . The degree is a nonnegative integer.

Consider a jacket \mathcal{J} of a $(D + 1)$ colored graph \mathcal{G} with $2k = |\mathcal{V}|$ vertices. The number of vertices and lines of \mathcal{J} are: $|\mathcal{V}(\mathcal{J})| = |\mathcal{V}| = 2k$ and $|\mathcal{E}(\mathcal{J})| = |\mathcal{E}| = (D + 1)k$, respectively. Hence, the number of faces of \mathcal{J} is $|\mathcal{F}(\mathcal{J})| = (D - 1)k + 2 - 2g(\mathcal{J})$. Taking into account that \mathcal{G} has $\frac{1}{2}D!$ jackets and each face belongs to $(D - 1)!$ jackets we obtain

$$F = |\mathcal{F}| = \frac{1}{(D - 1)!} \sum_{\mathcal{J}} |\mathcal{F}(\mathcal{J})| = \frac{D(D - 1)}{2} k + D - \frac{2}{(D - 1)!} \omega(\mathcal{G}). \quad (29)$$

This equation is crucial in establishing the universality results in the large N limit of random tensor models. Of course the same equation holds (replacing D by $D - 1$) for D -colored connected graphs. Note that, as F is an integer, $\omega(\mathcal{G})$ is a multiple of $\frac{2}{(D - 1)!}$.

We now consider the D -bubbles of \mathcal{G} with colors $\hat{0}$ (i.e. the connected subgraphs of \mathcal{G} with lines of colors $1, 2 \dots D$). We denote them $\mathcal{B}_{(\mu)}$. As they are D -colored graphs, they also possess jackets, which we denote by $\mathcal{J}_{(\mu)}^{\hat{0}}$. It is rather elementary to construct the jackets of the bubbles $\mathcal{J}_{(\mu)}^{\hat{0}}$ from the jackets of the graph \mathcal{J} [15, 16, 19]. Let us construct the ribbon graph $\mathcal{J}^{\hat{0}}$ consisting of vertex, edge and face sets:

$$\begin{aligned} \mathcal{V}(\mathcal{J}^{\hat{0}}) &= \mathcal{V}(\mathcal{J}) = \mathcal{V}, & \mathcal{E}(\mathcal{J}^{\hat{0}}) &= \mathcal{E}(\mathcal{J}) \setminus \mathcal{E}^0(\mathcal{J}) = \mathcal{E} \setminus \mathcal{E}^0, \\ \mathcal{F}(\mathcal{J}^{\hat{0}}) &= \left(\mathcal{F}(\mathcal{J}) \setminus \mathcal{F}^{(\tau^{-1}(0), 0)} \setminus \mathcal{F}^{(0, \tau(0))} \right) \cup \mathcal{F}^{(\tau^{-1}(0), \tau(0))}, \end{aligned} \quad (30)$$

that is having all the vertices of \mathcal{G} , all the lines of \mathcal{G} of colors different from 0 and some faces of \mathcal{G} . For instance, for the jacket corresponding to $(0, 1, \dots, D)$ the ribbon graph $\mathcal{J}^{\hat{0}}$ has faces $(12) \dots (D - 1D)$ and

¹It is however sometimes more transparent to over count the distinct jackets by a factor of two associating them one to one with cycles. For example, one can count that from the $D!$ cycles of $D + 1$ colors, $(D - 1)!$ will contain the pair ij and $(D - 1)!$ the pair ji .

²A moment of reflection reveals that the jackets necessarily represent orientable surfaces.

(D1). Given that the face set of \mathcal{J} is specified by a $(D+1)$ -cycle τ , the first thing to notice is that the face set of $\mathcal{J}^{\hat{0}}$ is specified by a D -cycle obtained from τ by deleting the color 0. The ribbon graph $\mathcal{J}^{\hat{0}}$ is the union of several connected components, $\mathcal{J}_{(\mu)}^{\hat{0}}$. Each $\mathcal{J}_{(\mu)}^{\hat{0}}$ is a jacket of a D -bubble $\mathcal{B}_{(\mu)}$. Conversely, every jacket of $\mathcal{B}_{(\mu)}$ is obtained from exactly D jackets of \mathcal{G}^3 .

Lemma 1. *Let \mathcal{G} be a closed connected $D+1$ colored graph and $\mathcal{B}_{(\mu)}$ its D -bubbles with colors $\hat{0}$. Then*

$$\omega(\mathcal{G}) \geq D \sum_{\mu} \omega(\mathcal{B}_{(\mu)}) . \quad (31)$$

As $\mathcal{J}_{(\mu)}^{\hat{0}}$ are in one-to-one correspondence with disjoint subgraphs of \mathcal{J} we have $g_{\mathcal{J}} \geq \sum_{\mu} g_{\mathcal{J}_{(\mu)}^{\hat{0}}}$. As every jacket $\mathcal{J}_{(\mu)}^{\hat{0}}$ is obtained as subgraph of exactly D distinct jackets \mathcal{J} , summing over all the jackets of \mathcal{G} proves the lemma (see [19] for more details).

Of particular importance in the sequel are the graphs \mathcal{G} of degree zero, $\omega(\mathcal{G}) = 0$. They have been extensively discussed in [29]. In $D \geq 3$, the $D+1$ colored graphs with degree zero have a very simple structure. A counting argument proves that such a graph must have at least a face with exactly two vertices. As all the jackets must be planar this in turn implies that the graph contains two vertices separated by exactly D lines. Albeit simple, the proof of the second statement is somewhat convoluted.

For $2+1$ colored graphs the degree equals the genus of the graph, hence the graphs of degree 0 are the planar graphs. For $D \geq 3$, the $D+1$ colored graphs of degree zero are called *melonics*.

Lemma 2. *Let $D \geq 3$. If \mathcal{G} is a closed connected $D+1$ colored graph of degree zero then \mathcal{G} has a face with exactly two vertices.*

Proof: Since \mathcal{G} is of degree zero it has $F = \frac{D(D-1)}{2}k + D$ faces, from equation (29). Denote F_s the number of faces with $2s$ vertices (every face must have an even number of vertices). Then

$$F_1 + F_2 + \sum_{s \geq 3} F_s = \frac{D(D-1)}{2}k + D . \quad (32)$$

Let $2k_{(\mu)}^{ij}$ be the number of vertices of the μ 'th face with colors ij . We count the total number of vertices by summing the numbers of vertices per face $\sum_{\mu, i < j} k_{(\mu)}^{ij} = F_1 + 2F_2 + \sum_{s \geq 3} s F_s = \frac{D(D+1)}{2}k$ (as each vertex contributes to $D(D+1)/2$ faces). Substituting F_2 from (32) we get

$$F_1 = 2D + \sum_{s \geq 3} (s-2)F_s + \frac{D(D-3)}{2}k . \quad (33)$$

Notice that on the right hand side, the first two terms yield a strictly positive contribution for any $D \geq 2$, whereas the third term changes sign when $D = 3$. □

This lemma explicitly breaks when $D = 2$: there exist planar graphs having no face with exactly two vertices. This is the deep origin of the fact that trace invariant measures can lead to non Gaussian matrices, but (as we will prove below) necessarily lead to Gaussian tensors in the large N limit.

Lemma 3. *If $D \geq 3$ and \mathcal{G} is a closed connected $D+1$ colored graph of degree zero, then it contains a D -bubble (i.e. subgraph with D colors) with exactly two vertices.*

We emphasize that the D lines of the D -bubble with two vertices can have *any* colors, $1, \dots, D$ but also $0, 2, \dots, D$ or $0, 1, 3, \dots, D$, etc.

Proof: From the previous lemma \mathcal{G} has a face (say of colors ij) with exactly two vertices (say v and \bar{v}). If, for all q , a unique line of color q connects v and \bar{v} we conclude. If the two lines of color q are different, $v \in l_1^q$,

³A jacket $\mathcal{J}_{(\mu)}^{\hat{0}}$ of $\mathcal{B}_{(\mu)}$ is specified by a D -cycle (missing the color 0). One can insert the color 0 anywhere along the cycle and thus get D independent $(D+1)$ -cycles.

$\bar{v} \in l_2^q$ we consider the jacket $\mathcal{J} = (\dots iqj \dots)$. It contains the faces (iq) and (qj) . As \mathcal{G} is of degree zero, \mathcal{J} is planar. As l_1^q and l_2^q separate the same two faces iq and qj , the graph \mathcal{J}' obtained from \mathcal{J} by deleting them has two lines less, but the same number of faces as \mathcal{J} . The Euler character of \mathcal{J}' is $\chi(\mathcal{J}') = \chi(\mathcal{J}) + 2 = 4$, hence \mathcal{J}' has two planar connected components. Then l_1^q and l_2^q separate a two point graph $\mathcal{G}^{(q)'} having at least two vertices less than \mathcal{G} . Note that $\mathcal{G}^{(q)'}$ is not a closed $D + 1$ colored graph, as it has two vertices which are not touched by lines of color q (the vertices on which l_1^q and l_2^q were hooked).$

We now take aside $\mathcal{G}^{(q)'}$. It can be transformed into a genuine closed $D + 1$ colored graph by adding a line l_{12}^q obtained by reconnecting l_1^q and l_2^q . Consider any jacket $\mathcal{J}_{\mathcal{G}^{(q)'}}$ of $\mathcal{G}^{(q)'}$. It is a planar ribbon graph with one external face. Adding l_{12}^q leads to a ribbon graph $\mathcal{J}_{\mathcal{G}^{(q)}}$ having one more line and one more face (the external face of $\mathcal{J}_{\mathcal{G}^{(q)'}}$ is divided into two faces by the new line l_{12}^q). It follows that $\mathcal{J}_{\mathcal{G}^{(q)}}$ is planar. The ribbon graph $\mathcal{J}_{\mathcal{G}^{(q)}}$ is one of the jackets of $\mathcal{G}^{(q)}$, hence $\omega(\mathcal{G}^{(q)}) = 0$.

Note that one can not naively iterate the argument, as the graph $\mathcal{G}^{(q)}$ has a line, l_{12}^q , which does not belong to \mathcal{G} . However, $\mathcal{G}^{(q)}$ has a face of colors $i'j'$ with exactly two vertices v', \bar{v}' . Again for all q' we consider the graph $\mathcal{G}^{(q,q')}$ obtained from $\mathcal{G}^{(q)}$ by reconnecting the two lines of color q' containing v', \bar{v}' into a unique line $l_{12}^{q'}$. The line l_{12}^q belongs to only one of these graphs, for a fixed q' . We then chose another one, say $\mathcal{G}^{(q,q'')}$ to iterate (if for all $q'' \neq q'$ the two vertices are connected by a unique line we obtained a bubble of \mathcal{G} with exactly two vertices and conclude). As l_{12}^q is not a line of $\mathcal{G}^{(q,q'')}$ but $l_{12}^{q'}$ is a line of $\mathcal{G}^{(q,q'')}$, all but one of the lines of $\mathcal{G}^{(q,q'')}$ belong to \mathcal{G} . We iterate until we reach a graph $\mathcal{G}^{(q,q'',\dots)}$ with exactly two vertices connected by $D + 1$ lines. Out of them D are lines of \mathcal{G} and form a D bubble. □

4.1.1 Melons

We call two vertices separated by D lines in a graph with $D + 1$ colors a **melon** (or an *internal* D -dipole, not to be confused with the D -dipole $\mathcal{B}^{(2)}$). We emphasize that a melon can have external legs of *any* color $0, 1$ up to D . The D internal lines of a melon with external lines of color i have colors $0, 1, \dots, i - 1, 1 + i, \dots, D$. Replacing a melon by a line corresponding to its external legs we obtain a graph of degree zero⁴ having two vertices less (and $\frac{D(D-1)}{2}$ less faces). Iterating, one reduces a graph of degree zero to a graph with exactly two vertices connected by $D + 1$ lines. Conversely all graphs of degree zero can be built by arbitrary insertions of melons on lines. The graphs of degree zero are then in one to one correspondence to colored rooted $D + 1$ -ary trees [24, 29].

First order. The lowest order graph consists in two vertices connected by $D + 1$ lines. We represent

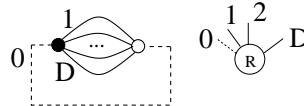


Figure 4: The first order melonic graph and its corresponding rooted tree.

this graph by the tree with one vertex decorated with $D + 1$ leaves. A *leaf* is a vertex with only one incident edge. The $D + 1$ leaves correspond to all the edges incident to the black vertex \bar{v} . (of course they are all also incident to the white vertex v). The leaves inherit the colors. This first vertex is called the *root vertex* (and is marked R). We consider all edges incident at the black vertex to be *active*. The leafs of the tree inherit this activity. See Figure 4 for an illustration.

Second order. At second order, $D + 1$ graphs contribute. They arise from inserting a melon (that is two vertices connected by D lines) on any of the $D + 1$ active lines of the first order graph. Say, we insert the new melon on the active edge of color 1. With respect to the new melon, all edges incident at its black vertex are deemed active, while the exterior edge (of color 1) incident at its white vertex is deemed inactive. This graph corresponds to a tree obtained from the first order tree by connecting its leaf of color 1 to a new $(D + 2)$ -valent vertex. This new vertex has $D + 1$ leaves, one of each color. The root and the new vertex are joined by tree line of color 1. The leaves correspond to the active lines (either of the root or on the

⁴Every jacket has two less vertices, $D + 1$ less lines and $D - 1$ less faces, hence its genus does not change.

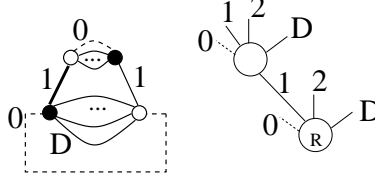


Figure 5: A second order melonic graph and its corresponding tree.

new melon). We presented this in figure 5. The inactive line of the graph (represented in bold in figure 5) corresponds to the tree line. All the active lines of the graph correspond to the leaves of the tree.

Order $k + 1$. We obtain the graphs at order $k + 1$ by inserting a melon on any of the active lines of a graph at order k . Once again, with respect to the new melon, all edges incident to its black vertex are deemed active, while the exterior edge incident to its white vertex is deemed inactive. In terms of the trees, we represent this insertion by connecting a $(D + 2)$ -valent vertex, with $D + 1$ active leaves, to one of the active leaves of a tree at order k . The new tree line inherits the color of this leaf.

The $2k$ vertices of the graph are in two to one correspondence to the k vertices of the tree. The $(D + 1)k$ lines of the graph are in one to one correspondence to the $(k - 1)$ lines and $Dk + 1$ leaves of the tree. The tree associated to a graph is a colored version of a Gallavotti-Nicolo tree [31].

If a graph is a $(D + 1)$ -colored melonic graph, all its subgraphs with D -colors (D -bubbles) are melonic. This is easy to see from the construction algorithm. Moreover, the D -ary trees of the D -bubbles with colors $\widehat{0}$, $\mathcal{B}_{(\mu)}$ are trivially obtained from the $(D + 1)$ -ary tree of the graph \mathcal{G} by deleting all lines and leaves of color 0.

We will use in the sequel the following two lemmas.

Lemma 4. *Let a melonic D -colored graph \mathcal{B} . Then there exists a unique melonic $D + 1$ colored graph \mathcal{G} with the same number of vertices which reduces to \mathcal{B} by deleting all the lines of color 0.*

The unique $D + 1$ -ary tree $\mathcal{T}_{\mathcal{G}}$ with k vertices which reduces to a given D -ary tree $\mathcal{T}_{\mathcal{B}}$ with k vertices by deleting all the tree lines and leaves of color 0 is the tree $\mathcal{T}_{\mathcal{B}}$ decorated by a leaf of color 0 on each of its vertices.

Lemma 5. *Let a melonic D -colored graph \mathcal{B} with $2k$ vertices. Then there exists a unique melonic $D + 1$ colored graph \mathcal{G} with $4k$ vertices which reduces to \mathcal{B} by deleting all the lines color 0, such that no two vertices of \mathcal{B} are connected (when seen as vertices in \mathcal{G}) by a line of color 0.*

As no two vertices of \mathcal{B} are connected (in \mathcal{G}) by a line of color zero, it follows that none of the tree vertices of the tree $\mathcal{T}_{\mathcal{B}}$ associated to \mathcal{B} (when seen as a subtree of the tree $\mathcal{T}_{\mathcal{G}}$ associated to \mathcal{G}) has a leaf of color 0. Therefore all the vertices in $\mathcal{T}_{\mathcal{B}}$ must be connected in $\mathcal{T}_{\mathcal{G}}$ to another vertex by a tree line of color 0. The tree $\mathcal{T}_{\mathcal{G}}$, obtained from $\mathcal{T}_{\mathcal{B}}$ by decorating each vertex with a line of color 0 (and a new end vertex), is unique and so is its associated graph \mathcal{G} with $4k$ vertices.

4.2 Open graphs and the boundary graph

We have discussed so far closed connected $D + 1$ colored graphs. We will now present open $D + 1$ colored graphs, that is graphs having some external lines.

Definition 9. A bipartite **open** $D + 1$ -colored graph is a graph $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ with vertex set $\mathcal{V}(\mathcal{G})$ and edge set $\mathcal{E}(\mathcal{G})$ such that:

- $\mathcal{V}(\mathcal{G})$ is bipartite, i.e. there exists a partition of the vertex set $\mathcal{V}(\mathcal{G}) = \mathcal{A}(\mathcal{G}) \cup \bar{\mathcal{A}}(\mathcal{G})$, such that for any element $l \in \mathcal{E}(\mathcal{G})$, then $l = (v, \bar{v})$ with $v \in \mathcal{A}(\mathcal{G})$ and $\bar{v} \in \bar{\mathcal{A}}(\mathcal{G})$. Their cardinalities satisfy $|\mathcal{V}(\mathcal{G})| = 2|\mathcal{A}(\mathcal{G})| = 2|\bar{\mathcal{A}}(\mathcal{G})|$.

- The positive (negative) vertices are of two types, **internal** vertices and **external** vertices, $\mathcal{A}(\mathcal{G}) = \mathcal{A}_{\text{int}}(\mathcal{G}) \cup \mathcal{A}_{\text{ext}}(\mathcal{G})$, $\bar{\mathcal{A}}(\mathcal{G}) = \bar{\mathcal{A}}_{\text{int}}(\mathcal{G}) \cup \bar{\mathcal{A}}_{\text{ext}}(\mathcal{G})$. The internal vertices are $D+1$ valent while the external vertices are 1-valent.
- The edge set is partitioned into D subsets $\mathcal{E}(\mathcal{B}) = \bigcup_{i=1}^D \mathcal{E}^i(\mathcal{B})$, where $\mathcal{E}^i(\mathcal{B}) = \{l^i = (v, \bar{v})\}$ is the subset of edges with color i . Furthermore the set of edges of color 0 is partitioned into **internal** and **external** lines of color 0, $\mathcal{E}^0(\mathcal{G}) = \mathcal{E}_{\text{int}}^0(\mathcal{G}) \cup \mathcal{E}_{\text{ext}}^0(\mathcal{G})$, such that the internal lines connect two internal vertices and the external lines connect an external and an internal vertex⁵. All the lines of color $i \neq 0$ are internal.
- The lines incident to a $D+1$ valent internal vertex have distinct colors, while the line incident to an external 1-valent vertex has color 0.

Some examples of open 3 + 1 colored graphs are presented on the left in figure 6. Both graphs have four external lines and four external vertices.

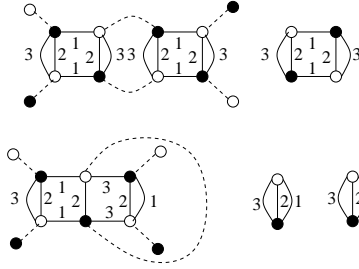


Figure 6: Open graphs and their boundary graphs.

The faces with colors $0i$ of an open $D+1$ colored graph (defined still as subgraphs with lines of colors 0 and i) fall in two categories: either they are **internal** faces, denoted $\mathcal{F}_{\text{int}}^{(0,i)}$ ($|\mathcal{F}_{\text{int}}^{(0,i)}| = F_{\text{int}}^{0i}$) i.e. they contain only internal lines, or they are **external** faces, denoted $\mathcal{F}_{\text{ext}}^{(0,i)}$ ($|\mathcal{F}_{\text{ext}}^{(0,i)}| = F_{\text{ext}}^{0i}$) i.e. they contain external lines of color 0.

The external faces $f \in \mathcal{F}_{\text{ext}}^{(0,i)}$ necessarily start and end on two external vertices u and \bar{u} , $f = (u, \bar{u})$. For every graph \mathcal{G} we build the **boundary graph** $\partial\mathcal{G}$ having a vertex u (resp. \bar{u}) for every external vertex of \mathcal{G} and a line of color i joining a u and a \bar{u} for every external face $f = (u, \bar{u}) \in \mathcal{F}_{\text{ext}}^{(0,i)}$ of \mathcal{G} . On the right in figure 6 we represented the boundary graphs $\partial\mathcal{G}$ of the two graphs \mathcal{G} . The boundary graph is a D colored graph and represents a tensor invariant, thus $\prod_{f=(u,\bar{u}) \in \cup_i \mathcal{F}_{\text{ext}}^{(0,i)}} \delta_{n_u^i \bar{n}_{\bar{u}}^i} = \delta_{n,\bar{n}}^{\partial\mathcal{G}}$.

Note that, as it is that case in the second example, in spite of the fact that \mathcal{G} itself is connected, the boundary graph $\partial\mathcal{G}$ can be disconnected.

4.3 Gaussian Distribution for Tensors

We now compute the large N trace invariant moments of the Gaussian distribution for a random tensor

$$\left\langle \text{Tr}_{\mathcal{B}}(\mathbb{T}, \bar{\mathbb{T}}) \right\rangle = \int \prod_{\bar{n}} \left(\frac{N^{D-1}}{\sigma^2} \frac{d\mathbb{T}_{\bar{n}} d\bar{\mathbb{T}}_{\bar{n}}}{2\pi i} \right) e^{-N^{D-1} \frac{1}{\sigma^2} \sum_{\bar{n}, \bar{\bar{n}}} \mathbb{T}_{\bar{n}} \delta_{\bar{n}, \bar{\bar{n}}} \bar{\mathbb{T}}_{\bar{\bar{n}}}} \text{Tr}_{\mathcal{B}}(\mathbb{T}, \bar{\mathbb{T}})}, \quad (34)$$

with the connected trace invariant operators

$$\text{Tr}_{\mathcal{B}}(\mathbb{T}, \bar{\mathbb{T}}) = \sum_{n, \bar{n}} \delta_{n\bar{n}}^{\mathcal{B}} \prod_{v, \bar{v} \in \mathcal{V}(\mathcal{B})} \mathbb{T}_{\bar{n}_v} \bar{\mathbb{T}}_{\bar{n}_{\bar{v}}}, \quad \delta_{n\bar{n}}^{\mathcal{B}} = \prod_{l^i = (v, \bar{v}) \in \mathcal{E}^i(\mathcal{B})} \delta_{n_v^i \bar{n}_{\bar{v}}^i}, \quad (35)$$

indexed by connected graphs \mathcal{B} with colors $1 \dots D$ having $2k(\mathcal{B})$ vertices (and $Dk(\mathcal{B})$ lines). Assume $\sigma = 1$. The number of faces of the D -colored graph associated to the observables computes from eq. (29) in terms

⁵Or two external vertices.

of its degree

$$\sum_{1 \leq i < j} F^{ij}(\mathcal{B}) = \frac{(D-1)(D-2)}{2} k(\mathcal{B}) + (D-1) - \frac{2}{(D-2)!} \omega(\mathcal{B}) . \quad (36)$$

The Gaussian expectation is a sum over contractions. As in the matrix case, we represent two tensors connected by a covariance as a dashed line to which we assign the color 0. We denote the full graph, including the color 0 by \mathcal{G} . An observable is a sum over graphs \mathcal{G} which restrict to \mathcal{B} by erasing the dashed lines of color 0. We already encounter such graphs in the case of matrices.

Definition 10. A $D+1$ colored graph \mathcal{G} is called a **covering graph** of \mathcal{B} if it reduces to \mathcal{B} by erasing the lines of color 0, $\mathcal{G} \setminus \mathcal{E}^0(\mathcal{G}) = \mathcal{B}$.

Every face of colors 0i in \mathcal{G} brings a free sum, hence a factor N . Every dashed line generated by the covariance brings a factor $\frac{1}{N^{D-1}}$. The moments of the Gaussian write

$$\begin{aligned} \left\langle \text{Tr}_{\mathcal{B}}(\mathbb{T}, \bar{\mathbb{T}}) \right\rangle &= \sum_{\mathcal{G}, \mathcal{G} \setminus \mathcal{E}^0(\mathcal{G}) = \mathcal{B}} N^{-k(\mathcal{B})(D-1)} N^{\sum_i F^{0i}(\mathcal{G})} \\ &= \sum_{\mathcal{G}, \mathcal{G} \setminus \mathcal{E}^0(\mathcal{G}) = \mathcal{B}} N^{-k(\mathcal{B})(D-1) + \sum_{0 \leq i < j} F^{ij}(\mathcal{G}) - \sum_{1 \leq i < j} F^{ij}(\mathcal{G})} , \end{aligned} \quad (37)$$

Note that $\sum_{0 \leq i < j} F^{ij}(\mathcal{G})$ is the total number of faces of the graph \mathcal{G} , while $\sum_{1 \leq i < j} F^{ij}(\mathcal{G}) = \sum_{1 \leq i < j} F^{ij}(\mathcal{B})$ is the number of faces of \mathcal{B} . Also we have $k(\mathcal{G}) = k(\mathcal{B})$. Using eq. (29) and (36), we get

$$\left\langle \text{Tr}_{\mathcal{B}}(\mathbb{T}, \bar{\mathbb{T}}) \right\rangle = \sum_{\mathcal{G}, \mathcal{G} \setminus \mathcal{E}^0(\mathcal{G}) = \mathcal{B}} N^{1 - \frac{2}{(D-1)!} \omega(\mathcal{G}) + \frac{2}{(D-2)!} \omega(\mathcal{B})} . \quad (38)$$

As both $\frac{2}{(D-1)!} \omega(\mathcal{G})$ and $\frac{2}{(D-2)!} \omega(\mathcal{B})$ are integers, the scaling with N of a graph \mathcal{G} contributing to the expectation of a trace invariant is always an integer. By Lemma 1, $\omega(\mathcal{G}) \geq D\omega(\mathcal{B})$, thus

$$1 - \frac{2}{(D-1)!} \omega(\mathcal{G}) + \frac{2}{(D-2)!} \omega(\mathcal{B}) = 1 - \frac{2}{D!} \omega(\mathcal{G}) - \frac{2}{D(D-2)!} [\omega(\mathcal{G}) - D\omega(\mathcal{B})] \leq 1 - \frac{2}{D!} \omega(\mathcal{G}) . \quad (39)$$

4.3.1 Melonic observables

From eq. (38) and (39) it follows that in the large N limit the expectation of an observable scales at most like N , and it scales like N only if there exists a melonic graph \mathcal{G} which restricts to \mathcal{B} by erasing the lines of color zero. This implies that \mathcal{B} itself must be melonic and, due to Lemma 4, it implies that \mathcal{G} is unique. The expectation of a melonic observable \mathcal{B} is therefore in the large N limit

$$\lim_{N \rightarrow \infty} N^{-1} \left\langle \text{Tr}_{\mathcal{B}}(\mathbb{T}, \bar{\mathbb{T}}) \right\rangle = 1 , \quad (40)$$

reproducing eq. (10) with $\Omega(\mathcal{B}) = 0$ and $R(\mathcal{B}) = 1$. Hence the melonic observables are the only observables of convergence order 0 in and their expectation at leading order is 1.

4.3.2 Arbitrary observables

Consider now a generic observable \mathcal{B} . The leading order contribution to eq. (38) is given by the covering graphs \mathcal{G} of \mathcal{B} having minimal degree.

Definition 11. A covering graph of \mathcal{B} of minimal degree \mathcal{G}^{\min} ,

$$\mathcal{G}^{\min} \setminus \mathcal{E}^0(\mathcal{G}^{\min}) = \mathcal{B} , \quad \text{with} \quad \omega(\mathcal{G}^{\min}) = \min_{\mathcal{G}, \mathcal{G} \setminus \mathcal{E}^0(\mathcal{G}) = \mathcal{B}} \omega(\mathcal{G}) , \quad (41)$$

is called a **minimal covering graph** of \mathcal{B} . Equivalently, the minimal covering graphs of \mathcal{B} are the covering graphs having the maximal possible number of faces $\sum_i F^{0i}(\mathcal{G})$.

Thus for all \mathcal{B} ,

$$\lim_{N \rightarrow \infty} N^{-1+\Omega(\mathcal{B})} \left\langle \text{Tr}_{\mathcal{B}}(\mathbb{T}, \bar{\mathbb{T}}) \right\rangle = R(\mathcal{B}) , \quad (42)$$

with the convergence order of the observable $\Omega(\mathcal{B}) = \frac{2}{(D-1)!} \omega(\mathcal{G}^{\min}) - \frac{2}{(D-2)!} \omega(\mathcal{B})$ and $R(\mathcal{B})$ the number of minimal covering graphs of \mathcal{B} . Take for example $D = 3$. Both invariants depicted in figure 7 are of order

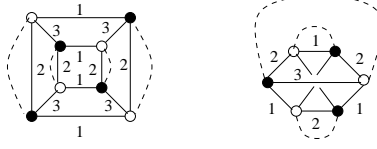


Figure 7: Observables of lower order for $D = 3$ and minimal covering graphs.

$\Omega(\mathcal{B}) = 1$ and the number of minimal covering graphs is in both cases $R(\mathcal{B}) = 3$. As already mentioned, for matrices ($D = 2$) the minimal covering graphs are exactly the planar graphs with one faces of colors 12. Note that lemma 4 can be reformulated as follows: for every melonic observable \mathcal{B} , there exists a unique minimal covering graph \mathcal{G} .

In general determining the degree of the minimal covering graphs (hence the order $\Omega(\mathcal{B})$ of an observable), and their number (hence $R(\mathcal{B})$) is a difficult problem. This is the reason for which here and below we prefer to treat the melonic observables and the rest of the observables separately. Indeed, in order to show that a tensor distributed with some μ_N converges in distribution to a Gaussian tensor we will show that for any observable one can establish a large N limit

$$\lim_{N \rightarrow \infty} N^{-1+\Omega(\mathcal{B})} \mu \left(\text{Tr}_{\mathcal{B}}(\mathbb{T}, \bar{\mathbb{T}}) \right) = R(\mathcal{B}) , \quad (43)$$

with $\Omega(\mathcal{B})$ and $R(\mathcal{B})$ identical with those of the Gaussian distribution eq. (42). However for most observable we will prove this indirectly, without actually computing neither $\Omega(\mathcal{B})$ nor $R(\mathcal{B})$. It is instructive then to see that for melonic observables one can establish by an alternate, direct route $\Omega(\mathcal{B}) = 0$ and $R(\mathcal{B}) = 1$ both for the i.i.d. and for the properly uniformly bounded trace invariant case.

We will need the following result.

Lemma 6. *Let \mathcal{G}^{\min} a minimal covering graph of the D colored graph \mathcal{B} with D odd (respectively even). Then any two lines of color 0 of \mathcal{G}^{\min} , $l_1^0 = (v, \bar{v})$, $l_2^0 = (w, \bar{w}) \in \mathcal{E}^0(\mathcal{G}^{\min})$ share **at most** $\frac{D-1}{2}$ (respectively $\frac{D}{2}$) faces of colors $0i$.*

Proof: Denote the number of faces of colors $0i$ shared by $l_1^0 = (v, \bar{v})$ and $l_2^0 = (w, \bar{w})$ by q . We build the open graph $\tilde{\mathcal{G}}^{\min}$ obtained from \mathcal{G}^{\min} by deleting the lines (v, \bar{v}) and (w, \bar{w}) and adding four external vertices \tilde{v} , \tilde{v} , \tilde{w} and \tilde{w} hooked to v , \bar{v} , w and \bar{w} respectively by external lines of color 0, as in figure 8.

The boundary graph of $\tilde{\mathcal{G}}^{\min}$, $\partial\tilde{\mathcal{G}}^{\min}$ is a D colored graph with four vertices. Hence it necessarily has the structure presented in figure 8 on the right, with q lines connecting \tilde{v} with \tilde{w} (respectively $\tilde{\bar{v}}$ and $\tilde{\bar{w}}$) and $D - q$ lines connecting \tilde{v} with $\tilde{\bar{v}}$ (respectively $\tilde{\bar{w}}$ and \tilde{w}).

Consider then the graph $\mathcal{G}^{\min, \times}$ obtained from \mathcal{G} by replacing the lines (v, \bar{v}) , (w, \bar{w}) by two new lines of color zero (v, \bar{w}) , (w, \bar{v}) , like in figure 8 on the second line. It is also a covering graph of \mathcal{B} .

The number of faces of colors $0i$ of \mathcal{G}^{\min} and $\mathcal{G}^{\min, \times}$ are respectively

$$\sum_i F^{0i}(\mathcal{G}^{\min}) = \sum_i F_{\text{int}}^{0i}(\tilde{\mathcal{G}}^{\min}) + q + 2(D - q) , \quad \sum_i F^{0i}(\mathcal{G}^{\min, \times}) = \sum_i F_{\text{int}}^{0i}(\tilde{\mathcal{G}}^{\min}) + D - q + 2q . \quad (44)$$

as \mathcal{G}^{\min} is a minimal covering graph of \mathcal{G} , we have

$$\sum_i F^{0i}(\mathcal{G}^{\min}) \geq \sum_i F^{0i}(\mathcal{G}^{\min, \times}) \Rightarrow 2q \leq D . \quad (45)$$

□

Remark that this lemma also holds for $D = 2$.

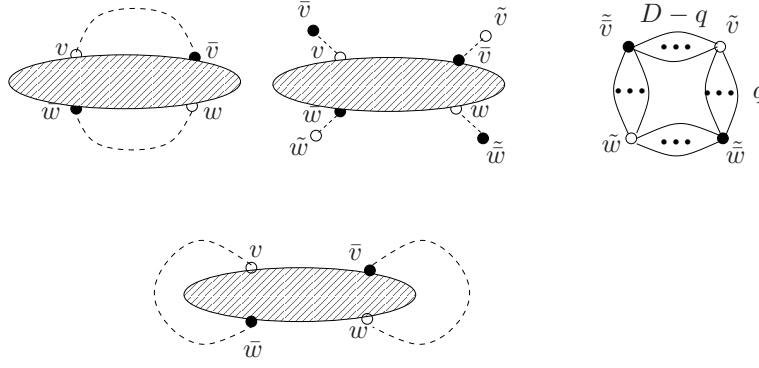


Figure 8: A minimal covering graph \mathcal{G}^{\min} , the opened graph $\tilde{\mathcal{G}}^{\min}$, the boundary graph $\partial\tilde{\mathcal{G}}^{\min}$ and $\mathcal{G}^{\min, \times}$

Lemma 7. *The convergence degree is a positive number. Moreover, for any \mathcal{B} , there exists an infinite family of graphs \mathcal{B}' such that $\Omega(\mathcal{B}) = \Omega(\mathcal{B}')$. Finally, the only normalization of the Gaussian such that both statements hold is the one of equation (34).*

Proof: From eq. (39), $\Omega(\mathcal{B}) \geq \frac{2}{D!} \omega(\mathcal{G}^{\min}) \geq 0$. Consider a graph \mathcal{B} and the graph \mathcal{B}' obtained by inserting a $D-1$ melon (say of colors $2, 3 \dots D$) on one of the lines (say of color 1) of \mathcal{B} . Call v and \bar{v} the vertices of this melon.

Consider a covering graph of \mathcal{B}' , \mathcal{G}' , such that the two vertices v and \bar{v} are connected by a line of color 0 in \mathcal{G}' . All minimal covering graphs of \mathcal{B}' are of this kind: any covering graph of \mathcal{B}' such that v and \bar{v} are not connected by a line of color 0 would have two lines of color 0 sharing $D-1$ faces, which is impossible by lemma 6 thus,

$$\Omega(\mathcal{B}') = \frac{2}{(D-1)!} \min_{\substack{\mathcal{G}', \mathcal{G}' \setminus \mathcal{E}^0(\mathcal{G}') = \mathcal{B}' \\ (v, \bar{v}) \in \mathcal{E}^0(\mathcal{G}')}} \omega(\mathcal{G}') - \frac{2}{(D-2)!} \omega(\mathcal{B}') \quad (46)$$

By reducing the melon v and \bar{v} , \mathcal{B} becomes \mathcal{B}' and \mathcal{G}' becomes some covering graph \mathcal{G} of \mathcal{B} . All covering graphs of \mathcal{B} can be obtained starting from some \mathcal{G}' of this kind. Moreover, as \mathcal{B} is obtained from \mathcal{B}' by reducing a $D-1$ dipole and \mathcal{G} from \mathcal{G}' by reducing a D dipole $\omega(\mathcal{B}') = \omega(\mathcal{B})$ and $\omega(\mathcal{G}') = \omega(\mathcal{G})$. Thus

$$\begin{aligned} \Omega(\mathcal{B}) &= \frac{2}{(D-1)!} \min_{\mathcal{G}, \mathcal{G} \setminus \mathcal{E}^0(\mathcal{G}) = \mathcal{B}} \omega(\mathcal{G}) - \frac{2}{(D-2)!} \omega(\mathcal{B}) \\ &= \frac{2}{(D-1)!} \min_{\substack{\mathcal{G}', \mathcal{G}' \setminus \mathcal{E}^0(\mathcal{G}') = \mathcal{B}' \\ (v, \bar{v}) \in \mathcal{E}^0(\mathcal{G}')}} \omega(\mathcal{G}') - \frac{2}{(D-2)!} \omega(\mathcal{B}') = \Omega(\mathcal{B}') . \end{aligned} \quad (47)$$

By inserting $D-1$ melons arbitrarily on the lines of \mathcal{B} one then builds an infinity of graphs \mathcal{B}' with $\Omega(\mathcal{B}') = \Omega(\mathcal{B})$. This proves the first part of the lemma.

For the second part, suppose that one choses a different normalization of the Gaussian measure

$$\left(\prod_{\vec{n}} N^\nu \frac{d\mathbb{T}_{\vec{n}} d\bar{\mathbb{T}}_{\vec{n}}}{2\pi i} \right) e^{-N^\nu \sum_{\vec{n}, \bar{\vec{n}}} \mathbb{T}_{\vec{n}} \delta_{\vec{n}, \bar{\vec{n}}} \bar{\mathbb{T}}_{\vec{n}}} . \quad (48)$$

Then eq. (38) becomes

$$N^{k(\mathcal{B})(D-1-\nu)+1-\frac{2}{(D-1)!}\omega(\mathcal{G})+\frac{2}{(D-2)!}\omega(\mathcal{B})} . \quad (49)$$

The order of convergence of an observable would then be

$$\Omega^{(\nu)}(\mathcal{B}) = \frac{2}{(D-1)!} \min_{\mathcal{G}, \mathcal{G} \setminus \mathcal{E}^0(\mathcal{G}) = \mathcal{B}} \omega(\mathcal{G}) - \frac{2}{(D-2)!} \omega(\mathcal{B}) + k(\mathcal{B})(\nu - (D-1)) , \quad (50)$$

which is positive for melonic \mathcal{B} only if $\nu \geq D - 1$. Moreover, if $\nu > D - 1$, then there exists only one observable with scaling $\nu - (D - 1)$, the D dipole itself. \square

Different scaling of the Gaussian can make sense, but only if one decides to look at **subsets** of observables. Consider for instance a tensor with 4 indices. One can decide to only consider tensor observables in which the tensor effectively acts as a $N^2 \times N^2$ matrix, that is the indices $(1, 2)$ and the indices $(3, 4)$ are always contracted between the same tensors. A scaling $\nu = 2$ leads to a well defined large N limit for these observables (this is just the usual large N limit of matrices). However other tensor observables do not behave well with this scaling: the melonic observables are arbitrarily divergent. The importance of the scaling N^{D-1} of the Gaussian is that it renders **all** the tensor observables convergent in the large N limit.

4.4 Proof of Theorem 1

The proof follows closely the one for matrices. Set the covariance of the atomic distribution to $\sigma^2 = 1$. Consider the observable associated to a graph \mathcal{B} with $2k$ vertices

$$\mu(\text{Tr}_{\mathcal{B}}(\mathbb{T}, \bar{\mathbb{T}})) = \frac{1}{N^{(D-1)k}} \sum_{n, \bar{n}} \delta_{n, \bar{n}}^{\mathcal{B}} \sum_{\pi} \kappa_{\pi} [T_{\bar{n}_1}, \bar{T}_{\bar{n}_1} \dots \bar{T}_{\bar{n}_k}] . \quad (51)$$

Again we represent all the second order moments as dashed lines of color 0. Again we deal with the higher order moments in a non canonical way, by representing them as dashed lines in some pairing of T and \bar{T} , but with further identifications one needs to track. The expectation writes as a sum over covering graphs of \mathcal{B} . The trace invariant operator composes with the identifications given by the cumulants and the faces $0i$ bring each a N . One obtains

$$\begin{aligned} \mu(\text{Tr}_{\mathcal{B}}(\mathbb{T}, \bar{\mathbb{T}})) &= \sum_{\mathcal{G}, \mathcal{G} \setminus \mathcal{E}^0(\mathcal{G}) = \mathcal{B}} N^{-k(D-1)} N^{\sum_i F^{0i}(\mathcal{G})} (\prod \delta) (\prod \kappa) \\ &= \sum_{\mathcal{G}, \mathcal{G} \setminus \mathcal{E}^0(\mathcal{G}) = \mathcal{B}} N^{-k(D-1) + \sum_{0 \leq i < j} F^{ij}(\mathcal{G}) - \sum_{0 < i < j} F^{ij}(\mathcal{G})} (\prod \delta) (\prod \kappa) \end{aligned} \quad (52)$$

where $(\prod \kappa)$ is a product over the cumulants associated to a graph. Note that if some of the lines in $\mathcal{E}^0(\mathcal{G})$ correspond to a higher order cumulant, all the indices of the faces $0i$ to which this lines belong are identified. We denote these extra identifications formally by the $(\prod \delta)$. These further identifications either play no role (if the indices of the faces $0i$ on the lines are already identified in \mathcal{G}), or they reduce the number of independent sums, hence the total scaling in N is strictly smaller than $-k(D - 1) + \sum_{0 \leq i < j} F^{ij}(\mathcal{G}) - \sum_{0 < i < j} F^{ij}(\mathcal{G})$. Again $\sum_{0 < i < j} F^{ij}(\mathcal{G}) = \sum_{0 < i < j} F^{ij}(\mathcal{B})$.

The scaling with N of a term in this sum is therefore at most

$$1 - \frac{2}{(D-1)!} \omega(\mathcal{G}) + \frac{2}{(D-2)!} \omega(\mathcal{B}) \leq 1 - \frac{2}{D!} \omega(\mathcal{G}) , \quad (53)$$

like in eq. (39). The graphs \mathcal{G} are covering graphs of \mathcal{B} , and the presence of a higher order cumulant (potentially) brings some extra constraints on the sums.

4.4.1 Melonic observables: direct computation

We first consider the case when the bound (53) is saturated. It follows that \mathcal{G} is a melonic graph, and consequently \mathcal{B} is melonic also.

Two lines of color 0 in a melonic graph \mathcal{G} having a unique bubble \mathcal{B} of colors $1 \dots D$ cannot share all their D faces of colors $0i$. Indeed \mathcal{G} is a covering graph of \mathcal{B} , and, as it has degree zero, it is also minimal, thus by lemma 6 any two lines can share at most $(D - 1)/2$ (or $D/2$) faces of colors $0i$. It follows that all the lines of color 0 of \mathcal{G} represent a second order cumulant (the presence of a higher order cumulant strictly decreases the scaling in N)

From lemma 4, \mathcal{G} is unique, thus at first order in N exactly one graph \mathcal{G} contributes and all lines of color 0 of \mathcal{G} represent a second order cumulant, hence

$$\lim_{N \rightarrow \infty} N^{-1+\Omega(\mathcal{B})} \mu\left(\text{Tr}_{\mathcal{B}}(\mathbb{T}, \bar{\mathbb{T}})\right) = R(\mathcal{B}) , \quad (54)$$

with $\Omega(\mathcal{B}) = 0$ and $R(\mathcal{B}) = 1$.

4.4.2 Arbitrary observables

We now deal with arbitrary observables. For all \mathcal{B} only the minimal covering graphs contribute at leading order to eq. (52)

$$\mu\left(\text{Tr}_{\mathcal{B}}(\mathbb{T}, \bar{\mathbb{T}})\right) = \sum_{\mathcal{G}^{\min}, \mathcal{G}^{\min} \setminus \mathcal{E}^0(\mathcal{G}) = \mathcal{B}} N^{-k(D-1)} N^{\sum_i F^{0i}(\mathcal{G})} (\prod \delta) (\prod \kappa) (1 + O(N^{-1})) \quad (55)$$

Again, the bound in eq. (53) is saturated (and consequently we get a contribution from the corresponding \mathcal{G}^{\min} above) only if the product $(\prod \delta)$ does not bring any extra identifications. Again, if \mathcal{G} is a minimal covering graph of \mathcal{B} , no two lines share all their faces, hence if \mathcal{G} possesses at least two lines coming from a higher order cumulant its contribution is strictly suppressed.

Hence the graphs contributing to eq. (55) are the minimal covering graphs of \mathcal{B} such that all their lines of color 0 correspond to a second order cumulant. Then

$$\lim_{N \rightarrow \infty} N^{-1+\Omega(\mathcal{B})} \mu\left(\text{Tr}_{\mathcal{B}}(\mathbb{T}, \bar{\mathbb{T}})\right) = R(\mathcal{B}) , \quad (56)$$

with $\Omega(\mathcal{B}) = \frac{2}{(D-1)!} \omega(\mathcal{G}^{\min}) - \frac{2}{(D-2)!} \omega(\mathcal{B})$ and $R(\mathcal{B})$ the number of minimal covering graphs of \mathcal{B} (as every minimal covering graph contributes exactly once), reproducing the moments of the Gaussian distribution. \square

4.5 Proof of Theorem 2

Following the discussion of the trace invariant measures for matrices, the expectation of an observable \mathcal{B} with $2k$ vertices for a trace invariant measure for tensors,

$$\mu\left(\text{Tr}_{\mathcal{B}}(\mathbb{T}, \bar{\mathbb{T}})\right) = \sum_{n, \bar{n}} \delta_{n\bar{n}}^{\mathcal{B}} \sum_{\pi} \kappa_{\pi} [\mathbb{T}_{\bar{n}_1}, \bar{\mathbb{T}}_{\bar{n}_1} \dots \bar{\mathbb{T}}_{\bar{n}_k}] , \quad (57)$$

writes as sums over doubled graphs $\mathcal{G} \supset \mathcal{B}$ generalizing (25)

$$\begin{aligned} \mu\left(\text{Tr}_{\mathcal{B}}(\mathbb{T}, \bar{\mathbb{T}})\right) &= \sum_{\mathcal{G} \supset \mathcal{B}, \mathcal{G} \setminus \mathcal{E}^0(\mathcal{G}) = \mathcal{B} \cup_{\alpha=1}^{|\alpha|} \left(\cup_{\rho=1}^{\rho(\mathcal{B}(\alpha))} \mathcal{B}_{\rho}(\alpha) \right)} N^{\left(-2(D-1)k + D|\alpha| - \sum_{\alpha=1}^{|\alpha|} \rho(\mathcal{B}(\alpha)) \right)} \\ &\quad \sum_{n, \bar{n}} \left(\prod_{i=1}^D \prod_{l^i = (v, \bar{v}) \in \mathcal{E}^i \left(\mathcal{B} \cup_{\alpha=1}^{|\alpha|} \left(\cup_{\rho=1}^{\rho(\mathcal{B}(\alpha))} \mathcal{B}_{\rho}(\alpha) \right) \right)} \delta_{n_v^i \bar{n}_{\bar{v}}^i} \right) \prod_{l^0 = (v, \bar{v}) \in \mathcal{E}^0(\mathcal{G})} \left(\prod_{i=1}^D \delta_{n_v^i \bar{n}_{\bar{v}}^i} \right) \\ &\quad \prod_{\alpha=1}^{|\alpha|} K(\mathcal{B}(\alpha), N) . \end{aligned} \quad (58)$$

Recall that the subgraphs with colors $1 \dots D$ of the doubled graph \mathcal{G} fall in two categories. One of them, \mathcal{B} (having no label α), corresponds to the initial observable, while the others $\mathcal{B}_{\rho}(\alpha)$ correspond to the various cumulants $\kappa_{2k(\alpha)}$. These graphs are connected by dashed lines of color 0 and, like for random matrices, the Kronecker δ 's compose along the faces with colors $0i$. The expectation of the observable writes as a sum over all doubled graphs which contain \mathcal{B}

$$\mu\left(\text{Tr}_{\mathcal{B}}(\mathbb{T}, \bar{\mathbb{T}})\right) = \sum_{\mathcal{G} \supset \mathcal{B}} N^{-2(D-1)k + D|\alpha| - \sum_{\alpha=1}^{|\alpha|} \rho(\mathcal{B}(\alpha)) + \sum_i F^{0i}(\mathcal{G})} \prod_{\alpha=1}^{|\alpha|} K(\mathcal{B}(\alpha), N) . \quad (59)$$

Using again the fact that the number of faces of colors $0i$ computes as the total number of faces minus the ones which don't have the color 0, the scaling with N computes further

$$-2(D-1)k + D|\alpha| - \sum_{\alpha=1}^{|\alpha|} \rho(\mathcal{B}(\alpha)) + \sum_{0 \leq i < j} F^{ij}(\mathcal{G}) - \sum_{0 < i < j} F^{ij}(\mathcal{G}), \quad (60)$$

taking into account that each face with colors ij , $0 < i < j$ belongs either to \mathcal{B} or to some $\mathcal{B}_\rho(\alpha)$, $\sum_{0 < i < j} F^{ij}(\mathcal{G}) = \sum_{0 < i < j} F^{ij}(\mathcal{B}) + \sum_{\alpha=1}^{|\alpha|} \sum_{\rho=1}^{\rho(\mathcal{B}(\alpha))} F^{ij}(\mathcal{B}_\rho(\alpha))$ the scaling computes to

$$\begin{aligned} & -2(D-1)k + D|\alpha| - \sum_{\alpha=1}^{|\alpha|} \rho(\mathcal{B}(\alpha)) + \left(\frac{D(D-1)}{2} k(\mathcal{G}) + D - \frac{2}{(D-1)!} \omega(\mathcal{G}) \right) \\ & - \left(\frac{(D-1)(D-2)}{2} k(\mathcal{B}) + D - 1 - \frac{2}{(D-2)!} \omega(\mathcal{B}) \right) \\ & - \sum_{\alpha=1}^{|\alpha|} \sum_{\rho=1}^{\rho(\mathcal{B}(\alpha))} \left(\frac{(D-1)(D-2)}{2} k(\mathcal{B}_\rho(\alpha)) + D - 1 - \frac{2}{(D-2)!} \omega(\mathcal{B}_\rho(\alpha)) \right). \end{aligned} \quad (61)$$

and recalling that the doubled graph \mathcal{G} has $4k$ vertices, $k(\mathcal{G}) = 2k$ while \mathcal{B} has k vertices, $k(\mathcal{B}) = k$ and $\sum_{\alpha=1}^{|\alpha|} \sum_{\rho=1}^{\rho(\mathcal{B}(\alpha))} k(\mathcal{B}_\rho(\alpha)) = k$ we obtain

$$\begin{aligned} \mu\left(\text{Tr}_{\mathcal{B}}(\mathbb{T}, \bar{\mathbb{T}})\right) &= \sum_{\mathcal{G} \supset \mathcal{B}} \prod_{\alpha=1}^{|\alpha|} K(\mathcal{B}(\alpha), N) \\ & N^{1 - \frac{2}{(D-1)!} \omega(\mathcal{G}) + \frac{2}{(D-2)!} \omega(\mathcal{B}) + \frac{2}{(D-2)!} \sum_{\alpha=1}^{|\alpha|} \sum_{\rho=1}^{\rho(\mathcal{B}(\alpha))} \omega(\mathcal{B}_\rho(\alpha)) - D \sum_{\alpha=1}^{|\alpha|} (\rho(\mathcal{B}(\alpha)) - 1)}. \end{aligned} \quad (62)$$

As \mathcal{B} and $\mathcal{B}_\rho(\alpha)$ are all the subgraphs (bubbles) of colors $\hat{0}$ of the graph \mathcal{G} , and using lemma 1, we bound the scaling with N of \mathcal{G} by

$$N^{1 - \frac{2}{D!} \omega(\mathcal{G}) - D \sum_{\alpha=1}^{|\alpha|} (\rho(\mathcal{B}(\alpha)) - 1)}. \quad (63)$$

4.5.1 Melonic observables: direct computation

Again we first discuss the case when the bound in eq. (63) is saturated. Then \mathcal{G} is a melonic graph such that every cumulant $\kappa_{2k(\alpha)}$ is represented by a unique connected invariant, $\rho(\mathcal{B}(\alpha)) = 1$.

As \mathcal{G} is melonic, \mathcal{B} must be melonic. Furthermore \mathcal{G} has $4k$ vertices, $2k$ of them belonging to \mathcal{B} and the other $2k$ to the invariants $\mathcal{B}_\rho(\alpha)$ (coming from the cumulants $\kappa_{2k(\alpha)}$), and all lines of color 0 connect some vertex in \mathcal{B} with a vertex belonging to one of the $\mathcal{B}_\rho(\alpha)$'s. By Lemma 5, \mathcal{G} is unique. Moreover its associated tree is the tree of \mathcal{B} with all vertices decorated by lines of color 0 ending in a tree vertex corresponding to some $\mathcal{B}_\rho(\alpha)$, hence all $\mathcal{B}_\rho(\alpha) = \mathcal{B}^{(2)}$. It follows that for melonic bubbles \mathcal{B}

$$\lim_{N \rightarrow \infty} N^{-1} \mu\left(\text{Tr}_{\mathcal{B}}(\mathbb{T}, \bar{\mathbb{T}})\right) = \left(\lim_{N \rightarrow \infty} K(\mathcal{B}^{(2)}, N) \right)^{k(\mathcal{B})}, \quad (64)$$

reproducing the expectation values of melonic observables of a Gaussian distribution of covariance $K(\mathcal{B}^{(2)}) = \lim_{N \rightarrow \infty} K(\mathcal{B}^{(2)}, N)$.

4.5.2 Arbitrary observables

Consider now an arbitrary observable \mathcal{B} . Note that if some of the connected components $\mathcal{B}_\rho(\alpha)$ come from the same cumulant ($\rho(\mathcal{B}(\alpha)) > 1$), the contribution of the doubled graph \mathcal{G} in eq. (62) is strictly suppressed

with respect to the one coming from the same doubled graph, but with all $\rho(\mathcal{B}(\alpha)) = 1$. Thus at leading order in N , we get

$$\mu\left(\text{Tr}_{\mathcal{B}}(\mathbb{T}, \bar{\mathbb{T}})\right) = \sum_{\mathcal{G} \supset \mathcal{B}} \prod_{\mathcal{B}_1(\alpha)} K(\mathcal{B}_1(\alpha), N) N^{1 - \frac{2}{(D-1)!} \omega(\mathcal{G}) + \frac{2}{(D-2)!} \omega(\mathcal{B}) + \frac{2}{(D-2)!} \sum_{\alpha=1}^{|\alpha|} \omega(\mathcal{B}_1(\alpha))}, \quad (65)$$

where $\mathcal{B}_1(\alpha)$ is the unique connected component of the graph representing the cumulant $\kappa_{2k(\alpha)}$.

Among the doubled graph $\mathcal{G} \supset \mathcal{B}$ contributing, some represent a minimal covering graph \mathcal{G}^{\min} of \mathcal{B} decorated by a two point cumulant on all the lines of color 0. We denote such a graph $\mathcal{G}^{\min} \cup_{\mathcal{E}^0(\mathcal{G}^{\min})} \mathcal{B}^{(2)}$. In this case every cumulant is a D -dipole $\mathcal{B}_1(\alpha) = \mathcal{B}^{(2)}$. We note that $\omega(\mathcal{B}^{(2)}) = 0$ (as the D -dipole is the first melonic graph with D colors). Moreover, $\mathcal{G}^{\min} \cup_{\mathcal{E}^0(\mathcal{G}^{\min})} \mathcal{B}^{(2)}$ has $\frac{D(D-1)}{2} k(\mathcal{G}^{\min})$ extra faces with respect to \mathcal{G}^{\min} (all the faces of colors $0 < i < j$ made by lines of the various $\mathcal{B}^{(2)}$ insertions) and $2k(\mathcal{G}^{\min})$ extra vertices (two vertices for each $\mathcal{B}^{(2)}$ insertion), hence by eq. (29) $\omega(\mathcal{G}^{\min} \cup_{\mathcal{E}^0(\mathcal{G}^{\min})} \mathcal{B}^{(2)}) = \omega(\mathcal{G}^{\min})$. Separating these terms among the terms contributing to the expectation we get

$$\mu\left(\text{Tr}_{\mathcal{B}}(\mathbb{T}, \bar{\mathbb{T}})\right) = \sum_{\mathcal{G}^{\min}, \mathcal{G}^{\min} \setminus \mathcal{E}^0(\mathcal{G}^{\min}) = \mathcal{B}} \left[K(\mathcal{B}^{(2)}, N) \right]^{k(\mathcal{B})} N^{1 - \frac{2}{(D-1)!} \omega(\mathcal{G}^{\min}) + \frac{2}{(D-2)!} \omega(\mathcal{B})} + \text{Rest} \quad (66)$$

therefore, provided that the rest of the terms are subleading in $1/N$, we get

$$\lim_{N \rightarrow \infty} N^{-1 + \Omega(\mathcal{B})} \mu\left(\text{Tr}_{\mathcal{B}}(\mathbb{T}, \bar{\mathbb{T}})\right) = \left[\lim_{N \rightarrow \infty} K(\mathcal{B}^{(2)}, N) \right]^{k(\mathcal{B})} R(\mathcal{B}), \quad (67)$$

with $\Omega(\mathcal{B}) = \frac{2}{(D-1)!} \omega(\mathcal{G}^{\min}) - \frac{2}{(D-2)!} \omega(\mathcal{B})$ and $R(\mathcal{B})$ the number of minimal covering graphs of \mathcal{B} , reproducing large N moments of the Gaussian distribution of covariance $K(\mathcal{B}^{(2)}) = \lim_{N \rightarrow \infty} K(\mathcal{B}^{(2)}, N)$.

To conclude we now prove that all the other terms contributing to the expectation are strictly suppressed in $1/N$. Among these terms some represent non minimal covering graphs of \mathcal{B} decorated by insertions of $\mathcal{B}^{(2)}$ on all lines of color 0, $\mathcal{G}^{\text{n. min}} \cup_{\mathcal{E}^0(\mathcal{G}^{\text{n. min}})} \mathcal{B}^{(2)}$. For such graphs eq. (66) holds, with $\omega(\mathcal{G}^{\min})$ replaced by $\omega(\mathcal{G}^{\text{n. min}}) > \omega(\mathcal{G}^{\min})$, hence they are suppressed.

Consider now that \mathcal{G} has at least a higher order cumulant $\mathcal{B}_1(\alpha) \neq \mathcal{B}^{(2)}$. The scaling with N of \mathcal{G} is, from eq. (59) (recall that $\rho(\mathcal{B}(\alpha)) = 1$),

$$-2(D-1)k + (D-1)|\alpha| + \sum_i F^{0i}(\mathcal{G}). \quad (68)$$

Consider two lines of color 0, (v, \bar{a}) and (a, \bar{v}) touching two vertices v and \bar{v} separated by a line of color j of $\mathcal{B}_1(\alpha)$. We will compare the scaling of \mathcal{G} with the one of the graph $\tilde{\mathcal{G}}$ obtained by reconnecting the two lines of color 0 into a line of color 0, namely (a, \bar{a}) , with a $\mathcal{B}^{(2)}$ insertion, and reconnecting all the other lines touching v and \bar{v} respecting the colors⁶ (see figure 9). We consider the two point subgraph $\mathcal{B}^{(2)}$ as coming from a different cumulant in $\tilde{\mathcal{G}}$.

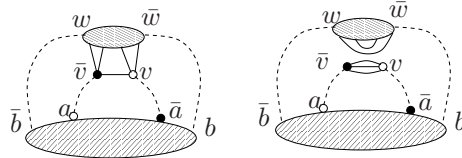


Figure 9: The graphs \mathcal{G} and $\tilde{\mathcal{G}}$.

⁶ If there are several lines (of colors different from 0) connecting v and \bar{v} in \mathcal{G} , we delete them. If $\mathcal{B}_1(\alpha)$ divides in several connected components under this procedure, we associate a different label α to each of them (i.e. we consider each of them as coming from a different cumulant in $\tilde{\mathcal{G}}$). Both these cases give strictly subleading contributions.

The graph $\tilde{\mathcal{G}}$ is also a doubled graph $\tilde{\mathcal{G}} \supset \mathcal{B}$, having $|\tilde{\alpha}| = |\alpha| + 1$, and $F^{0i}(\tilde{\mathcal{G}}) \geq F^{0i}(\mathcal{G}) - (D - 1)$ (as the face of colors $0j$ is not affected by this change, and all the other $D - 1$ faces $0q$ touching v and \bar{v} can at most merge two by two), thus

$$-2(D - 1)k + (D - 1)|\alpha| + \sum_i F^{0i}(\mathcal{G}) \leq -2(D - 1)k + (D - 1)|\tilde{\alpha}| + \sum_i F^{0i}(\tilde{\mathcal{G}}) \quad (69)$$

and equality holds only if all the faces of colors $0q$, for all $q \neq j$ touching v and \bar{v} are merged after this reduction. Iterating we reduce the order of all cumulants and obtain a doubled graph representing a covering graph $\mathcal{G}^{\text{final}}$ of \mathcal{B} with two point insertions $\mathcal{B}^{(2)}$ on all lines, $\mathcal{G}^{\text{final}} \cup_{\mathcal{E}^0(\mathcal{G}^{\text{final}})} \mathcal{B}^{(2)}$.

At the last step we reduced a four point cumulant connected to the rest of the graph by four lines color 0 namely (v, \bar{a}) , (a, \bar{v}) and another two lines, say (b, \bar{w}) and (w, \bar{b}) . In order for eq. (69) to hold with an $=$ sign (if not the contribution of \mathcal{G} is strictly suppressed with respect to the one of $\mathcal{G}^{\text{final}} \cup_{\mathcal{E}^0(\mathcal{G}^{\text{final}})} \mathcal{B}^{(2)}$), it follows that the two lines of $\mathcal{G}^{\text{final}}$, (a, \bar{a}) and (b, \bar{b}) (obtained after eliminating the insertions $\mathcal{B}^{(2)}$ in $\mathcal{G}^{\text{final}} \cup_{\mathcal{E}^0(\mathcal{G}^{\text{final}})} \mathcal{B}^{(2)}$) share all the $D - 1$ faces of colors $0q$ for $q \neq j$. Hence from lemma 6 the graph $\mathcal{G}^{\text{final}}$ can not be minimal. In all cases the contribution of \mathcal{G} is strictly suppressed with respect to the one of minimal covering graphs decorated by $\mathcal{B}^{(2)}$ insertions thus eq. (67) always holds. \square

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A Perturbed Gaussian measures

In this lengthy appendix we discuss in some detail a large class of probability measures for tensors to which our results apply: the perturbed Gaussian measures. These measures appear naturally in physics and describe random D dimensional triangulations [24, 19]. A perturbed Gaussian measure is defined by an action

$$S(\mathbb{T}, \bar{\mathbb{T}}) = \sum_{\vec{n}} \mathbb{T}_{\vec{n}} \delta_{\vec{n}\bar{\vec{n}}} \bar{\mathbb{T}}_{\bar{\vec{n}}} + \sum_{\mathcal{H}} t_{\mathcal{H}} \text{Tr}_{\mathcal{H}}(\mathbb{T}, \bar{\mathbb{T}}) \\ d\mu = \frac{1}{Z(t_{\mathcal{H}}, N)} \left(\prod_{\vec{n}} N^{D-1} \frac{d\mathbb{T}_{\vec{n}} d\bar{\mathbb{T}}_{\bar{\vec{n}}}}{2\pi i} \right) e^{-N^{D-1} S(\mathbb{T}, \bar{\mathbb{T}})}, \quad (70)$$

with $Z(t_{\mathcal{H}}, N)$ a normalization constant. We consider only the case when all \mathcal{H} are connected graphs with D colors, hence the most general “single trace” model.

There are two levels of precision at which one can study the perturbed Gaussian measures. The first one is the *perturbative* level. Perturbed Gaussian measures are evaluated in terms of Feynman graphs (and the reader is assumed in this appendix to have some familiarity with them). The perturbative level establishes bounds on individual graphs. It is at this level that we work in section A.1. The relation between the Feynman graphs and the doubled graphs used so far hides a number of subtleties. The aim of the first part of this appendix is to first detail this relation and second to prove that the joint cumulants of such a distribution respect the proper uniform bounds in eq. (7) at the *perturbative level*, i.e. for each Feynman graph contributing to a cumulant. This result is insufficient to claim that such measures are properly uniformly bounded, but it constitutes a good indication that they might be.

The second level at which one must treat the perturbed Gaussian measures is the *non perturbative* or *constructive* level in which one must promote the perturbative bounds to constructive bounds valid for the full cumulants which are sums over graphs. In order to establish proper uniform boundedness at the constructive level one must resum the perturbation theory. This is notoriously difficult, as the perturbation series is not summable (the number of graphs grows too fast) but only Borel summable (to be precise Borel -

Le Roy of an order fixed by the maximal degree monomial in the perturbation of the Gaussian measure). The resummation of the perturbative series requires a set of techniques quite different from the ones employed in the rest of this paper, amounting to a research field in itself: constructive field theory [25]. We will therefore treat in the second part of this appendix at the constructive level only a particular example of a perturbed Gaussian measure. The techniques we present here (generalizing the one introduced in [33] to tensors) should be further refined along the lines of [34] to establishing proper uniform boundedness of all the cumulants for any polynomially perturbed Gaussian measure.

A.1 Perturbative bounds

We will show in the first part of this appendix that the cumulants of this measure respect *in perturbation theory*

$$\kappa_{2k}[\mathbb{T}_{\vec{n}_1}, \bar{\mathbb{T}}_{\vec{n}_1} \dots \mathbb{T}_{\vec{n}_k}, \bar{\mathbb{T}}_{\vec{n}_k}] = \sum_{\substack{\mathcal{B} = \cup_{\rho=1}^{\rho(\mathcal{B})} \mathcal{B}_\rho \\ k(\mathcal{B})=k}} N^{-2(D-1)k(\mathcal{B})+D-\rho(\mathcal{B})} K(\mathcal{B}, N) \prod_{\rho=1}^{\rho(\mathcal{B})} \delta_{n\bar{n}}^{\mathcal{B}_\rho}, \quad (71)$$

where the sum runs over all D -colored graphs \mathcal{B} with $2k$ vertices and $\rho(\mathcal{B})$ connected components \mathcal{B}_ρ , and $\lim_{N \rightarrow \infty} K(\mathcal{B}, N)$ exists and is finite for all \mathcal{B} . By theorem 2 *all* the perturbed Gaussian measures become Gaussian in the large N limit in the perturbative sense. It is however naive to conclude that the large N limit of such models is trivial. The covariance of the large N Gaussian (equaling the large N expectation of the D -dipole observable $\mathcal{B}^{(2)}$) is the *full* resummed two point function of the model, and has a very non trivial dependence on the parameters $t_{\mathcal{H}}$ [20].

We will evaluate the moments and cumulants of the measure (70) by expanding in Taylor series with respect to $t_{\mathcal{H}}$ (hence the name “perturbative results”). As we will see in the second part of this appendix the ensuing series are not summable, and in order to promote these results to the non perturbative level one needs to work quite a lot. The joint moments of the probability distribution of tensor entries

$$\mu(\mathbb{T}_{\vec{n}_1}, \bar{\mathbb{T}}_{\vec{n}_1} \dots \mathbb{T}_{\vec{n}_k}, \bar{\mathbb{T}}_{\vec{n}_k}) = \int d\mu \mathbb{T}_{\vec{n}_1} \bar{\mathbb{T}}_{\vec{n}_1} \dots \mathbb{T}_{\vec{n}_k} \bar{\mathbb{T}}_{\vec{n}_k}, \quad (72)$$

are expressed as sums over Feynman graphs. They are obtained as follows: upon expanding with respect to $t_{\mathcal{H}}$, each invariant $\text{Tr}_{\mathcal{H}}(\mathbb{T}, \bar{\mathbb{T}})$ (represented by a graph \mathcal{H} with D colors $1, 2 \dots D$) becomes an insertion in a Gaussian integral. The Gaussian integral is then evaluated in terms of contractions (pairings) of tensor entries. For each such contraction scheme one draws a Feynman graph. The invariants act as *effective vertices* (interactions) of the Feynman graphs (not to be confused with the black and white vertices of \mathcal{H} itself which represent the tensor entries \mathbb{T} and $\bar{\mathbb{T}}$). The effective interactions are supplemented by *effective lines*, (propagators, Wick contractions) representing the pairing of two tensors $\mathbb{T}_{\vec{n}}$ and $\bar{\mathbb{T}}_{\vec{n}}$ with the Gaussian measure. We represent the contraction of two tensor as a dashed lines of color 0 connecting the corresponding black and white vertices. Thus a Feynman graph \mathcal{G} has $D + 1$ colors, 0 for the dashed lines and $1 \dots D$ for the effective interactions.

The insertions $\mathbb{T}_{\vec{n}_1}, \bar{\mathbb{T}}_{\vec{n}_1} \dots \mathbb{T}_{\vec{n}_k}, \bar{\mathbb{T}}_{\vec{n}_k}$ in the joint moment are represented as external black or white vertices of valence 1. The external vertices are joined by lines of color 0 to the rest of the Feynman graph. Thus the dashed lines of color 0 fall into two categories: *internal* joining two tensors (that is black and white vertices) on two effective interactions \mathcal{H} and \mathcal{H}' and *external* joining an external vertex with an internal vertex on some \mathcal{H} . The Feynman graphs are then are nothing but open $D + 1$ colored graphs of definition 9. Two examples of Feynman graphs are presented on the left in figure 6. The effective interactions \mathcal{H} are represented with solid lines of colors 1, 2 and 3. Both graphs have four external lines of color 0.

The cumulants $\kappa_{2k}(\mathbb{T}_{\vec{n}_1}, \bar{\mathbb{T}}_{\vec{n}_1} \dots \mathbb{T}_{\vec{n}_k}, \bar{\mathbb{T}}_{\vec{n}_k})$ are sums over *connected* Feynman graphs \mathcal{G} with $2k$ external (univalent) vertices. We stress that the \mathcal{G} 's contributing to a cumulant are connected as a graph with $D + 1$ colors.

Each $D + 1$ colored graph represents an abstract D dimensional simplicial pseudo manifold [17]. This pseudo manifold is obtained by associating a D -simplex to each (black and white) vertex in the graph (hence to each tensor entry \mathbb{T} and $\bar{\mathbb{T}}$). The $D - 1$ simplices bounding the D simplex are colored 0, 1 up to D . This induces colorings on all lower dimensional simplices. The D simplices are then glued respecting all

the colorings: a line in the graph represents the unique gluing of two D simplices along boundary $D - 1$ simplices which respects the colorings of the $D - 1$, $D - 2$ etc. simplices. An effective operator $\text{Tr}_{\mathcal{H}}(\mathbb{T}, \bar{\mathbb{T}})$ with $2k$ tensor represents the gluing of $2k$ simplices around a vertex forming a “chunk”. For example in three dimensions an operator represents a gluing of tetrahedra around a vertex. The boundary of such a chunk is paved by triangles (represented by the half lines of color 0). Topologically the chunks are cones over their boundary, hence they can have non trivial topology. A Feynman graph represents the gluing of such chunks into a pseudo manifold. As the combinatorial weights and amplitudes of the graphs are fixed by the Feynman rules, the measures (70) encode a canonical theory of random pseudo manifolds in arbitrary dimensions. The leading order melonic graphs represent spheres.

Each contraction in the Gaussian integral (hence dashed line of color 0) replaces the two tensors $\mathbb{T}_{\vec{n}}$ and $\bar{\mathbb{T}}_{\vec{p}}$ by a covariance $\frac{1}{N^{D-1}} \delta_{\vec{n}\vec{p}}$. The amplitude of a Feynman graph is then

$$\begin{aligned} A^{\mathcal{G}} &= \sum_{n, \bar{n}} \left(\prod_{\mathcal{H}(\mathcal{G})} t_{\mathcal{H}(\mathcal{G})} N^{D-1} \delta_{n\bar{n}}^{\mathcal{H}(\mathcal{G})} \right) \left(\prod_{l^0=(v, \bar{v}) \in \mathcal{E}^0(\mathcal{G})} \frac{1}{N^{D-1}} \prod_{i=1}^D \delta_{n_v^i \bar{n}_{\bar{v}}^i} \right) \\ &= \left(\prod_{\mathcal{H}(\mathcal{G})} t_{\mathcal{H}(\mathcal{G})} \right) N^{(D-1)H(\mathcal{G}) - (D-1)|\mathcal{E}^0(\mathcal{G})|} \\ &\quad \sum_{n, \bar{n}} \left(\prod_{\mathcal{H}(\mathcal{G})} \prod_{l^i=(v, \bar{v}) \in \mathcal{E}^i(\mathcal{H}(\mathcal{G}))} \delta_{n_v^i \bar{n}_{\bar{v}}^i} \right) \left(\prod_{l^0=(v, \bar{v}) \in \mathcal{E}^0(\mathcal{G})} \prod_{i=1}^D \delta_{n_v^i \bar{n}_{\bar{v}}^i} \right), \end{aligned} \quad (73)$$

where $\mathcal{H}(\mathcal{G})$ runs over all the subgraphs with colors $1 \dots D$ of \mathcal{G} , $H(\mathcal{G}) = |\mathcal{H}(\mathcal{G})|$ denotes the number of such subgraphs and $|\mathcal{E}^0(\mathcal{G})|$ the number of lines of color 0 of \mathcal{G} . The Kronecker δ 's compose along the faces with colors $0i$. The faces with colors $0i$ of \mathcal{G} are either internal or external (see section 4.2). The internal faces $\mathcal{F}_{\text{int}}^{(0,i)}(\mathcal{G})$ (with $F_{\text{int}}^{0i}(\mathcal{G}) = |\mathcal{F}_{\text{int}}^{(0,i)}(\mathcal{G})|$) yield a free sum hence bring a large factor N . The external faces $f \in \mathcal{F}_{\text{ext}}^{(0,i)}$ necessarily start and end on two external vertices u and \bar{u} corresponding to two external insertions \mathbb{T} and $\bar{\mathbb{T}}$ in the joint moment, $f = (u, \bar{u})$. The amplitude of a graph becomes

$$A^{\mathcal{G}} = \left(\prod_{\mathcal{H}(\mathcal{G})} t_{\mathcal{H}(\mathcal{G})} \right) N^{(D-1)H(\mathcal{G}) - (D-1)|\mathcal{E}^0(\mathcal{G})| + \sum_i F_{\text{int}}^{0i}(\mathcal{G})} \prod_{f=(u, \bar{u}) \in \cup_i \mathcal{F}_{\text{ext}}^{0i}(\mathcal{G})} \delta_{n_u^i \bar{n}_{\bar{u}}^i}, \quad (74)$$

and the operator $\prod_{f=(u, \bar{u}) \in \cup_i \mathcal{F}_{\text{ext}}^{0i}(\mathcal{G})} \delta_{n_u^i \bar{n}_{\bar{u}}^i}$ reproduces the trace invariant operator associated to the boundary graph $\partial\mathcal{G}$ (see again section 4.2). We finally conclude that the amplitude of a graph is

$$A^{\mathcal{G}} = \delta_{n\bar{n}}^{\partial\mathcal{G}} N^{(D-1)H(\mathcal{G}) - (D-1)|\mathcal{E}^0(\mathcal{G})| + \sum_i F_{\text{int}}^{0i}(\mathcal{G})} \left(\prod_{\mathcal{H}(\mathcal{G})} t_{\mathcal{H}(\mathcal{G})} \right). \quad (75)$$

As already mentioned (the second example in figure 6) in spite of the fact that \mathcal{G} itself is connected, the boundary graph $\partial\mathcal{G}$ can be disconnected. It follows that a cumulant, which is a sum over connected graphs \mathcal{G} expands as a sum over all possible D colored graphs (connected or not) corresponding to the possible boundary graphs $\mathcal{B} = \partial\mathcal{G}$

$$\kappa(\mathbb{T}_{\vec{n}_1}, \bar{\mathbb{T}}_{\vec{n}_1} \dots \bar{\mathbb{T}}_{\vec{n}_k}) = \sum_{\mathcal{B} = \cup_{\rho=1}^{\rho(\mathcal{B})} \mathcal{B}_{\rho}} \left[\sum_{\mathcal{G}, \partial\mathcal{G}=\mathcal{B}} \frac{1}{S(\mathcal{G})} \left(\prod_{\mathcal{H}(\mathcal{G})} t_{\mathcal{H}(\mathcal{G})} \right) N^{(D-1)H(\mathcal{G}) - (D-1)|\mathcal{E}^0(\mathcal{G})| + \sum_i F_{\text{int}}^{0i}(\mathcal{G})} \right], \quad (76)$$

where \mathcal{B} runs over all closed D colored graphs with $2k$ vertices (connected or not), and \mathcal{G} runs over all $D + 1$ colored connected graphs with $2k$ external legs of color 0 whose boundary is \mathcal{B} . For every \mathcal{G} , $\mathcal{H}(\mathcal{G})$ runs over its internal subgraphs with colors 1, 2 up to D , $|\mathcal{E}^0(\mathcal{G})|$, $F_{\text{int}}^{0i}(\mathcal{G})$ and $H(\mathcal{G})$ denote the total number of lines of color 0 (including the $2k$ external lines), internal faces of colors $0i$ of \mathcal{G} , and respectively subgraphs \mathcal{H} . Finally, $S(\mathcal{G})$ is some symmetry factor.

We can now describe the precise relationship between the Feynman graphs and the doubled graphs used to establish theorem 2. The doubled graphs for a perturbed Gaussian measure consist in the observable \mathcal{B} and the *boundary graphs* $\mathcal{B}_{\rho}(\alpha)$, $\rho = 1, \dots, \rho(\mathcal{B}(\alpha))$ of the various Feynman graphs $\mathcal{G}(\alpha)$ contributing to each of the cumulants $\kappa_{2k(\alpha)}$ arising in an expansion in cumulants of the moment $\mu(\text{Tr}_{\mathcal{B}}(\mathbb{T}, \bar{\mathbb{T}}))$.

In order to conclude *at the perturbative level* that all perturbed Gaussian measures become Gaussian in the large N limit, we must prove that for all \mathcal{G} with $\partial\mathcal{G} = \mathcal{B} = \cup_{\rho=1}^{\rho(\mathcal{B})} \mathcal{B}_\rho$

$$N^{(D-1)H(\mathcal{G}) - (D-1)|\mathcal{E}^0(\mathcal{G})| + \sum_i F_{\text{int}}^{0i}(\mathcal{G})} \leq N^{-2(D-1)k(\partial\mathcal{G}) + D - \rho(\partial\mathcal{G})}, \quad (77)$$

that is the cumulants respect eq. (71).

Theorem 4. *For every connected $D+1$ colored graph \mathcal{G} with $2k$ external vertices, $|\mathcal{E}^0(\mathcal{G})|$ lines of color 0, $F_{\text{int}}^{0i}(\mathcal{G})$ internal faces of colors $0i$, $H(\mathcal{G})$ subgraphs with colors $1, \dots, D$ and $\rho(\partial\mathcal{G})$ connected components of the boundary graph $\partial\mathcal{G}$*

$$(D-1)H(\mathcal{G}) - (D-1)|\mathcal{E}^0(\mathcal{G})| + \sum_i F_{\text{int}}^{0i}(\mathcal{G}) \leq -2(D-1)k(\partial\mathcal{G}) + D - \rho(\partial\mathcal{G}). \quad (78)$$

Proof. The proof of this theorem is divided into two parts. We first present an iterative algorithm which reduces the graph \mathcal{G} to the $D+1$ colored graph $\partial\mathcal{G} \cup \mathcal{E}_{\text{ext}}^0(\mathcal{G})$ consisting in the D colored graph $\partial\mathcal{G}$ decorated by an external line of color 0 for each of its $2k$ vertices. At each step of this algorithm we will obtain a graph $\mathcal{G}^{(s)}$ interpolating between $\mathcal{G}^{(0)} = \mathcal{G}$ and $\mathcal{G}^{(s_{\text{max}})} = \partial\mathcal{G} \cup \mathcal{E}_{\text{ext}}^0(\mathcal{G})$. Second we will prove that at each step of this algorithm the quantity

$$Q(s) = D - C(\mathcal{G}^{(s)}) + (D-1)[H(\mathcal{G}^{(s)}) - C(\mathcal{G}^{(s)})] - (D-1)|\mathcal{E}^0(\mathcal{G}^{(s)})| + \sum_i F_{\text{int}}^{0i}(\mathcal{G}^{(s)}), \quad (79)$$

with $C(\mathcal{G}^{(s)})$ the numbers of connected components of $\mathcal{G}^{(s)}$, $H(\mathcal{G}^{(s)})$ the number of bubbles (subgraphs) with colors $1, 2$ up to D of $\mathcal{G}^{(s)}$, $|\mathcal{E}^0(\mathcal{G}^{(s)})|$ the number of lines of color 0 of $\mathcal{G}^{(s)}$ and $F_{\text{int}}^{0i}(\mathcal{G}^{(s)})$ the number of internal faces of colors $0i$ of $\mathcal{G}^{(s)}$ is strictly increasing. As $Q(0) = (D-1)H(\mathcal{G}) - (D-1)|\mathcal{E}^0(\mathcal{G})| + \sum_i F_{\text{int}}^{0i}(\mathcal{G})$ and $Q(s_{\text{max}}) = D - \rho(\partial\mathcal{G}) - 2(D-1)k(\partial\mathcal{G})$, we conclude.

Obtaining $\partial\mathcal{G} \cup \mathcal{E}_{\text{ext}}^0(\mathcal{G})$. The algorithm we present here has been introduced in [32].

Consider a connected $D+1$ colored graph $\mathcal{G}^{(s)}$ with $2k$ external vertices. We first define an *internal $q+1$ dipole* with colors $0, i_1, \dots, i_q$ as two internal vertices v and \bar{v} of $\mathcal{G}^{(s)}$ connected by an *internal* line of color 0 and exactly q lines of colors i_1, \dots, i_q . An example of an internal $q+1$ dipole with colors $0, 1, \dots, q$ is given on the left in figure 10. An internal $q+1$ dipole can be *contracted*. The contraction consist in deleting the two vertices v and \bar{v} and the $q+1$ lines connecting them, and reconnecting the remaining lines respecting the colors.

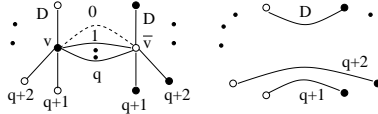


Figure 10: A $q+1$ dipole with colors $0, 1, \dots, q$.

Under a contraction we obtain a new graph $\mathcal{G}^{(s+1)}$ having two less vertices, one less internal line of color 0, q less internal faces of colors $0i$ and the same number of external vertices, $2k$. Note that the new graph, $\mathcal{G}^{(s+1)}$, can potentially be disconnected. Note also that neither the external vertices of $\mathcal{G}^{(s)}$, nor its internal vertices hooked by a line of color 0 to external vertices can be deleted.

Consider the graph obtained starting from \mathcal{G} and contracting iteratively, in an arbitrary order, the maximal number of internal $q+1$ dipoles with colors $0, i_1, \dots, i_q$. The number of internal dipoles contracted equals the number of internal lines of color 0 of \mathcal{G} , $|\mathcal{E}_{\text{int}}^0(\mathcal{G})|$. We show below that the final graph $\mathcal{G}^{(s_{\text{max}})}$ is $\partial\mathcal{G} \cup \mathcal{E}_{\text{ext}}^0(\mathcal{G})$, the boundary graph of \mathcal{G} decorated by an external line of color 0 on each of its vertices. Examples of this full reduction are given in figure 11.

The final graph $\mathcal{G}^{(s_{\text{max}})}$ has $4k$ vertices, $2k$ coinciding with the external vertices of \mathcal{G} and $2k$ with the internal vertices of \mathcal{G} hooked to external vertices by external lines of color 0. It has no more internal lines of color 0 but still has $2k$ external lines of color 0. As the internal vertices are each touched by exactly one line for every color $1, 2$ up to D , $\mathcal{G}^{(s_{\text{max}})}$ has exactly k lines of every color $1, 2$ up to D . Furthermore $\mathcal{G}^{(s_{\text{max}})}$

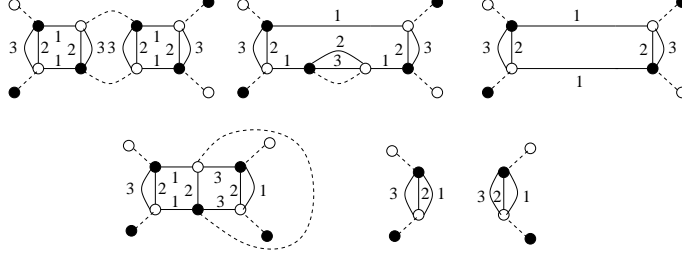


Figure 11: The reduction of all the internal dipoles in a graph.

has no internal faces of colors $0i$. However the external faces with colors $0i$ can never be deleted by this procedure hence all the faces of colors $0i$ of $\mathcal{G}^{(s_{\max})}$ are external and they are one to one to the Dk external faces of colors $0i$ of \mathcal{G} . It follows that all (external) faces $0i$ of $\mathcal{G}^{(s_{\max})}$ contain exactly one line of color i , connecting the two internal vertices hooked to the external vertices which share the face $0i$. The lines of color 0 of $\mathcal{G}^{(s_{\max})}$ are all external lines and one to one to the external lines of \mathcal{G} , $\mathcal{E}^0(\mathcal{G}^{(s_{\max})}) = \mathcal{E}_{\text{ext}}^0(\mathcal{G})$. By deleting the lines of color 0 (and flipping the black and white vertices), the final graph $\mathcal{G}^{(s_{\max})} \setminus \mathcal{E}^0(\mathcal{G}^{(s_{\max})})$ will have a vertex for every external point of \mathcal{G} , and a line of color i connecting two vertices u and \bar{u} for every external face $f = (u, \bar{u})$ of colors $0i$ of \mathcal{G} . Hence $\mathcal{G}^{(s_{\max})} \setminus \mathcal{E}^0(\mathcal{G}^{(s_{\max})}) = \partial\mathcal{G}$.

Bounds. Suppose we reduce a dipole of colors $0, 1, \dots, q$ to pass from $\mathcal{G}^{(s)}$ to $\mathcal{G}^{(s+1)}$. We have two cases. Either the two vertices v and \bar{v} belong to two different bubbles (connected components) with colors $1, 2$ up to D and the dipole is necessarily a 1 dipole made exclusively by a line of color 0, or the two vertices belong to the same bubble with colors $1, 2$ up to D .

First case. We have $v \in \mathcal{H}_1$ and $\bar{v} \in \mathcal{H}_2$, and both \mathcal{H}_1 and \mathcal{H}_2 belong to the same connected component of $\mathcal{G}^{(s)}$. The number of connected components does not change by contracting the dipole, $C(\mathcal{G}^{(s+1)}) = C(\mathcal{G}^{(s)})$. To see this, consider the bubble \mathcal{H}_1 . As it is a graph with D colors it can not become disconnected by deleting v . Chose a spanning tree T_1 in $\mathcal{H}_1 \setminus v$ (the bubble with v omitted), and a tree T_2 in $\mathcal{H}_2 \setminus \bar{v}$. Complete it by adding the lines of color 1 touching $v \in l_v^1$ and $\bar{v} \in l_{\bar{v}}^1$ and the line of color $l_{v\bar{v}}^0 = (v, \bar{v})$, and finally to a tree in the entire connected component of $\mathcal{G}^{(s)}$ by adding lines T_{rest} . The tree $T_1 \cup l_v^1 \cup l_{v\bar{v}}^0 \cup l_{\bar{v}}^1 \cup T_2 \cup T_{\text{rest}}$ becomes after reduction the tree $T_1 \cup l^1 \cup T_2 \cup T_{\text{rest}}$ (with l^1 the new line of color 1), spanning one connected component in $\mathcal{G}^{(s+1)}$.

The two bubbles $\mathcal{H}_1, \mathcal{H}_2 \subset \mathcal{G}^{(s)}$ are collapsed into a unique bubble of $\mathcal{G}^{(s+1)}$ thus $H(\mathcal{G}^{(s+1)}) = H(\mathcal{G}^{(s)}) - 1$. The number of lines of color 0 decreases by 1, $|\mathcal{E}^0(\mathcal{G}^{(s+1)})| = |\mathcal{E}^0(\mathcal{G}^{(s)})| - 1$, and the number of internal faces of color $0i$ does not change $F_{\text{int}}^{0i}(\mathcal{G}^{(s+1)}) = F_{\text{int}}^{0i}(\mathcal{G}^{(s)})$ hence

$$\begin{aligned} D - C(\mathcal{G}^{(s+1)}) + (D-1)[H(\mathcal{G}^{(s+1)}) - C(\mathcal{G}^{(s+1)})] - (D-1)|\mathcal{E}^0(\mathcal{G}^{(s+1)})| + \sum_i F_{\text{int}}^{0i}(\mathcal{G}^{(s+1)}) \\ = D - C(\mathcal{G}^{(s)}) + (D-1)[H(\mathcal{G}^{(s)}) - C(\mathcal{G}^{(s)})] - (D-1)|\mathcal{E}^0(\mathcal{G}^{(s)})| + \sum_i F_{\text{int}}^{0i}(\mathcal{G}^{(s)}) . \end{aligned} \quad (80)$$

Second case. Both v and \bar{v} belong to the same bubble $v, \bar{v} \in \mathcal{H}$. In this case the number of connected components of $\mathcal{G}^{(s)}$ can increase when contracting the $q+1$ -dipole (note that, like in the previous case q can be zero, but it can also be larger than 0 in this case). As each of the new lines $D-q$ lines (one for each color not belonging to the $q+1$ dipole) must belong to some connected component of $\mathcal{G}^{(s+1)}$, we have $C(\mathcal{G}^{(s+1)}) - C(\mathcal{G}^{(s)}) \leq D-q-1$. Moreover, if one of these lines belongs to a connected component created by the contraction, then it certainly belongs to a new bubble of colors $1, 2$ up to D created by this contraction. Hence $C(\mathcal{G}^{(s+1)}) - C(\mathcal{G}^{(s)}) \leq H(\mathcal{G}^{(s+1)}) - H(\mathcal{G}^{(s)})$. As before, $|\mathcal{E}^0(\mathcal{G}^{(s+1)})| = |\mathcal{E}^0(\mathcal{G}^{(s)})| - 1$, but q internal faces of colors $0i$ are deleted, $\sum_i F_{\text{int}}^{0i}(\mathcal{G}^{(s+1)}) = \sum_i F_{\text{int}}^{0i}(\mathcal{G}^{(s)}) - q$, hence

$$\begin{aligned} D - C(\mathcal{G}^{(s+1)}) + (D-1)[H(\mathcal{G}^{(s+1)}) - C(\mathcal{G}^{(s+1)})] - (D-1)|\mathcal{E}^0(\mathcal{G}^{(s+1)})| + \sum_i F_{\text{int}}^{0i}(\mathcal{G}^{(s+1)}) \\ \geq D - C(\mathcal{G}^{(s)}) - (D-q-1) \\ + (D-1)[H(\mathcal{G}^{(s)}) - C(\mathcal{G}^{(s)})] \end{aligned}$$

$$-(D-1)|\mathcal{E}^0(\mathcal{G}^{(s)})| + D - 1 + \sum_i F_{\text{int}}^{0i}(\mathcal{G}^{(s)}) - q, \quad (81)$$

thus in both cases $Q(s+1) \geq Q(s)$. \square

A.2 From perturbative to constructive bounds

In order to prove that the full resummed cumulant is properly uniformly bounded one must perform a good deal of extra work. Before proceeding to the non perturbative treatment, we first establish a technical lemma.

Lemma 8. *Let σ and τ and ξ be three permutations of k elements. We denote $c(\tau)$ the number of cycles of the permutation τ . Then*

$$c(\xi) + c(\sigma\xi) + c(\tau) + c(\sigma\tau^{-1}) \leq 2 + 2k. \quad (82)$$

Proof: To triple of permutations ξ , σ and τ we associate a ribbon graph constructed as follows.

Consider the decomposition of ξ in cycles, $\xi = C_1 \dots C_{c(\xi)}$, each of length $|C_r|$. We draw a fat vertex for every cycle of ξ having $4|C_r|$ halflines. We assign a label $l_q \alpha_q \beta_q j_q, l_{\xi(q)} \alpha_{\xi(q)} \beta_{\xi(q)} j_{\xi(q)}, \dots$ turning clockwise, to each half line, see figure 12. Two consecutive halflines share a strand. The strands are of three kinds: solid (connecting j_q to $l_{\xi(q)}$), dashed (connecting α_q with β_q) and wiggly (connecting l_q to α_q or β_q to j_q). Thus the halflines representing l 's and j 's are solid-wiggly, and the ones representing α 's and β 's are wiggly-dashed. We represent the permutations σ and τ by ribbon lines connecting the halflines l_q to $j_{\sigma(q)}$ and α_q with $\beta_{\tau(q)}$. The ribbon lines $l_q \rightarrow j_{\sigma(q)}$ are then solid-wiggly and the ribbon lines $\alpha_q \rightarrow \beta_{\tau(q)}$ are wiggly-dashed.

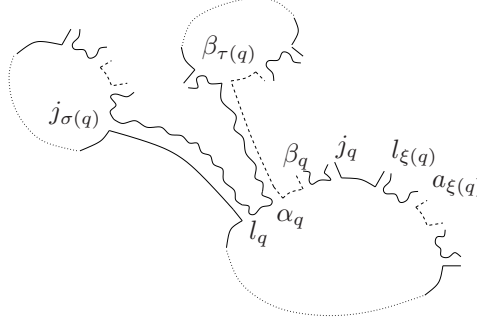


Figure 12: Ribbon graph associated to three permutations ξ , σ and τ .

The faces (closed strands) of the ribbon graph thus obtained are of three types:

- dashed faces $\beta_q \rightarrow \alpha_q \rightarrow \beta_{\tau(q)} \dots$. We call them “ τ faces” as they are indexed by the permutation τ . Their number is the number of cycles $c(\tau)$.
- solid faces $j_q \rightarrow l_{\xi(q)} \rightarrow j_{\sigma\xi(q)} \dots$. We call them “ $\sigma\xi$ ” as they are indexed by the permutation $\sigma\xi$. Their number is the number of cycles $c(\sigma\xi)$.
- wiggly faces $\beta_q \rightarrow \alpha_{\tau^{-1}(q)} \rightarrow l_{\tau^{-1}(q)} \rightarrow j_{\sigma\tau^{-1}(q)} \dots$. We call them “ $\sigma\tau^{-1}$ ” faces as they are indexed by the permutation $\sigma\tau^{-1}$. Their number is the number of cycles $c(\sigma\tau^{-1})$.

The ribbon graph has $c(\xi)$ vertices and $2k$ lines, hence

$$c(\xi) - 2k + c(\sigma\xi) + c(\tau) + c(\sigma\tau^{-1}) \leq 2. \quad (83)$$

\square

We are now going to the core of the non perturbative definition of the simplest quartically perturbed Gaussian measure. We will denote the indices either as \vec{n} or as $n\vec{\alpha}$ with $n = n^1$ and $\vec{\alpha} = n^2, \dots, n^D$. We will deal with the measure

$$S(\mathbb{T}, \bar{\mathbb{T}}) = \sum_{n\vec{\alpha}} \mathbb{T}_{n\vec{\alpha}} \delta_{n\vec{n}} \delta_{\vec{\alpha}\vec{\alpha}} \bar{\mathbb{T}}_{\vec{n}\vec{\alpha}} + \lambda \sum_{n\vec{\alpha}} \mathbb{T}_{n\vec{\alpha}} \bar{\mathbb{T}}_{\vec{p}\vec{\alpha}} \mathbb{T}_{\vec{p}\vec{\beta}} \bar{\mathbb{T}}_{\vec{n}\vec{\beta}} \delta_{n\vec{n}} \delta_{\vec{\alpha}\vec{\alpha}} \delta_{\vec{p}\vec{p}} \delta_{\vec{\beta}\vec{\beta}}$$

$$d\mu = \frac{1}{Z(\lambda, N)} \left(\prod_{n\vec{\alpha}} N^{D-1} \frac{d\mathbb{T}_{\vec{n}} d\bar{\mathbb{T}}_{\vec{n}}}{2\pi i} \right) e^{-N^{D-1} S(\mathbb{T}, \bar{\mathbb{T}})}, \quad (84)$$

The quartic perturbation corresponds to a melonic invariant. From now on all the repeated indices are summed. We consider the generating function of the moments of our distribution

$$Z(J, \bar{J}; \lambda, N) = \int \left(\prod_{n\vec{\alpha}} N^{D-1} \frac{d\mathbb{T}_{\vec{n}} d\bar{\mathbb{T}}_{\vec{n}}}{2\pi i} \right) e^{-N^{D-1} \left(S(\mathbb{T}, \bar{\mathbb{T}}) - \bar{\mathbb{T}}_{\vec{n}\vec{\beta}} J_{\vec{n}\vec{\beta}} - \mathbb{T}_{\vec{n}\vec{\beta}} \bar{J}_{\vec{n}\vec{\beta}} \right)}. \quad (85)$$

For every finite N , $Z(J, \bar{J}; \lambda, N)$ is defined by an integral over N^D complex variables which is absolutely convergent (and bounded by $e^{N^{D-1} J_{\vec{n}\vec{\beta}} \bar{J}_{\vec{n}\vec{\beta}} \delta_{\vec{n}\vec{n}} \delta_{\vec{\beta}\vec{\beta}}}$) for $\Re \lambda \geq 0$ and divergent for $\Re \lambda < 0$. Thus $Z(J, \bar{J}; \lambda, N)$ is an analytic function in the right half complex plane. The generating function of the cumulants (connected moments) is $W(J, \bar{J}; \lambda, N) = \ln Z(J, \bar{J}; \lambda, N)$,

$$\kappa(\mathbb{T}_{n_1 \vec{\alpha}_1}, \bar{\mathbb{T}}_{\bar{n}_1 \vec{\alpha}_1}, \dots, \mathbb{T}_{n_k \vec{\alpha}_k}, \bar{\mathbb{T}}_{\bar{n}_k \vec{\alpha}_k}) = N^{-2k(D-1)} \frac{\partial^{(2k)}}{\partial \bar{J}_{\bar{n}_1 \vec{\alpha}_1} \partial J_{n_1 \vec{\alpha}_1} \dots \partial \bar{J}_{\bar{n}_k \vec{\alpha}_k} \partial J_{n_k \vec{\alpha}_k}} W(J, \bar{J}; \lambda, N) \Big|_{J=\bar{J}=0}. \quad (86)$$

The perturbative treatment described in the previous section consists in two steps. To compute the cumulants (connected moments) one first takes the logarithm of $Z(J, \bar{J}; \lambda, N)$ and second one expands this logarithm in Taylor series around the point $\lambda = 0$. Both steps are in fact problematic. While one has a well controlled expression for $Z(J, \bar{J}; \lambda, N)$ as an absolutely convergent integral, the same does not hold for $W(J, \bar{J}; \lambda, N)$. Furthermore, $\lambda = 0$ belongs to the boundary of the analyticity domain of $Z(J, \bar{J}; \lambda, N)$. So far we don't know anything about the analyticity domain of $W(J, \bar{J}; \lambda, N)$, but $\lambda = 0$ can certainly **not be** in its interior: it can at best belong to its boundary. A Taylor expansion around a point belonging to the boundary of the analyticity domain of some function typically leads to series which are not summable, but only Borel summable.

One needs to find an intermediate expansion of $W(J, \bar{J}; \lambda, N)$ (hence a proper way to take the logarithm) which yields a definition of $W(J, \bar{J}; \lambda, N)$ in some analyticity domain, such that $\lambda = 0$ belongs to its boundary and furthermore $W(J, \bar{J}; \lambda, N)$ is the Borel sum of its Taylor expansion around $\lambda = 0$. Moreover, as we are interested in the $N \rightarrow \infty$ limit, both the convergence of the constructive expansion in its analyticity domain and the Borel summability around $\lambda = 0$ must be uniform in N .

This intermediate expansion is called the *constructive* expansion. The constructive expansion captures some features of the perturbative expansion (for instance the perturbative bounds on Feynman graphs can be promoted to bounds on the terms in the constructive expansion) but unlike the former it is absolutely convergent and provides **the good** non perturbative definition of the connected moments of perturbed Gaussian measures.

The draw back of the constructive expansion is that it is rather involved and its study requires a number of new techniques. Note that a priori one can give several constructive expansions of the same measure. In particular the constructive Loop Vertex Expansion (LVE) introduced in [33] has proved (the only one) adapted to matrix models. We will extend in the sequel the LVE to the random tensor model with measure given by eq. (84).

A.2.1 The Loop Vertex Expansion

The LVE is a combination of three ingredients: the Hubbard Stratonovich intermediate field representation, the universal Brydges-Kennedy-Abdesselam-Rivasseau forest formula and a replica trick. The last two ingredients are a recurrent feature of any constructive expansion while the first one is specific to the LVE.

Step 1: We use the Hubbard Stratonovich intermediate field representation of the interaction term

$$e^{-N^{D-1} \lambda \mathbb{T}_{n\vec{\alpha}} \bar{\mathbb{T}}_{\bar{n}\vec{\alpha}} \mathbb{T}_{p\vec{\beta}} \bar{\mathbb{T}}_{\bar{p}\vec{\beta}} \delta_{n\bar{n}} \delta_{\vec{\alpha}\vec{\alpha}} \delta_{p\bar{p}} \delta_{\vec{\beta}\vec{\beta}}} = \int \left(\prod_{ab} \frac{d\sigma_{ab} d\bar{\sigma}_{ab}}{2\pi i} \right) e^{-\sigma_{ab} \bar{\sigma}_{ab} - \sqrt{\lambda} N^{\frac{D-1}{2}} \mathbb{T}_{b\vec{\beta}} \bar{\mathbb{T}}_{\bar{b}\vec{\beta}} \delta_{\vec{\beta}\vec{\beta}} \delta_{\bar{n}a} \sigma_{ab} + \sqrt{\lambda} N^{\frac{D-1}{2}} \mathbb{T}_{a\vec{\alpha}} \bar{\mathbb{T}}_{\bar{a}\vec{\alpha}} \delta_{\vec{\alpha}\vec{\alpha}} \delta_{\bar{p}b} \bar{\sigma}_{ab}}, \quad (87)$$

where σ_{ab} is an auxiliary $N \times N$ matrix field. Denoting \mathbb{I} the identity matrix of size $N^{D-1} \times N^{D-1}$ we write more compactly

$$\int \left(\prod_{ab} \frac{d\sigma_{ab} d\bar{\sigma}_{ab}}{2\pi i} \right) e^{-\sigma_{ab} \bar{\sigma}_{ab} - N^{D-1} \bar{\mathbb{T}}_{\bar{n}\bar{\beta}} \left(\sqrt{\frac{\lambda}{N^{D-1}}} \sigma \otimes \mathbb{I} - \sqrt{\frac{\lambda}{N^{D-1}}} \sigma^\dagger \otimes \mathbb{I} \right)_{\bar{n}\bar{\beta}; p\bar{\alpha}} \mathbb{T}_{p\bar{\alpha}}} \quad (88)$$

thus $Z(J, \bar{J}; \lambda, N)$ becomes, denoting 1 the identity matrix of size $N \times N$,

$$Z(J, \bar{J}; \lambda, N) = \int \left(\prod_{n\bar{\alpha}} N^{D-1} \frac{d\mathbb{T}_{n\bar{\alpha}} d\bar{\mathbb{T}}_{n\bar{\alpha}}}{2\pi i} \right) \left(\prod_{ab} \frac{d\sigma_{ab} d\bar{\sigma}_{ab}}{2\pi i} \right) e^{-\sigma_{ab} \bar{\sigma}_{ab} - N^{D-1} \bar{\mathbb{T}}_{\bar{n}\bar{\beta}} \left(1 \otimes \mathbb{I} + \sqrt{\frac{\lambda}{N^{D-1}}} (\sigma - \sigma^\dagger) \otimes \mathbb{I} \right)_{\bar{n}\bar{\beta}; p\bar{\alpha}} \mathbb{T}_{p\bar{\alpha}} + N^{D-1} \bar{\mathbb{T}}_{\bar{n}\bar{\beta}} J_{n\bar{\beta}} + N^{D-1} \mathbb{T}_{n\bar{\beta}} \bar{J}_{n\bar{\beta}}} . \quad (89)$$

As $\sigma - \sigma^\dagger$ is antihermitian, $1 + \sqrt{\frac{\lambda}{N^{D-1}}} (\sigma - \sigma^\dagger)$ is invertible. We call its inverse the *resolvent* $R(\sigma) = \left[1 + \sqrt{\frac{\lambda}{N^{D-1}}} (\sigma - \sigma^\dagger) \right]^{-1}$ and note the following properties

$$\begin{aligned} \frac{\partial}{\partial \sigma_{np}} R(\sigma)_{ab} &= -\sqrt{\frac{\lambda}{N^{D-1}}} R(\sigma)_{an} R(\sigma)_{pb} , & \frac{\partial}{\partial \sigma_{np}^\dagger} R(\sigma)_{ab} &= \sqrt{\frac{\lambda}{N^{D-1}}} R(\sigma)_{an} R(\sigma)_{pb} , \\ \frac{\partial}{\partial \sigma_{np}} \text{tr} \ln(R(\sigma)) &= -\sqrt{\frac{\lambda}{N^{D-1}}} R_{pn} , & \frac{\partial}{\partial \sigma_{np}^\dagger} \text{tr} \ln(R(\sigma)) &= \sqrt{\frac{\lambda}{N^{D-1}}} R_{pn} , \\ \|R(\sigma)\| &\leq 1 \quad \forall \lambda \geq 0 , \end{aligned} \quad (90)$$

where $\|R(\sigma)\|$ denotes the operator norm. The introduction of the intermediate field renders the integration over $\mathbb{T}\bar{\mathbb{T}}$ Gaussian, thus

$$\begin{aligned} Z(J, \bar{J}; \lambda, N) &= \int \left(\prod_{\bar{n}p} \frac{d\sigma_{\bar{n}p} d\bar{\sigma}_{\bar{n}p}}{2\pi i} \right) e^{-\text{tr} \sigma \sigma^\dagger + \text{Tr} \ln(R(\sigma) \otimes \mathbb{I}) + N^{D-1} \bar{J}_{n\bar{\beta}} (R(\sigma) \otimes \mathbb{I})_{n\bar{\beta}; \bar{p}\bar{\alpha}} J_{\bar{p}\bar{\alpha}}} \\ &= \int \left(\prod_{\bar{n}p} \frac{d\sigma_{\bar{n}p} d\bar{\sigma}_{\bar{n}p}}{2\pi i} \right) e^{-\text{tr} \sigma \sigma^\dagger + N^{D-1} \text{tr} \ln(R(\sigma)) + N^{D-1} \text{tr}(R(\sigma) J J^\dagger)} , \end{aligned} \quad (91)$$

where tr denotes the trace over an index of size N , Tr denotes a trace over an index of size N^D , and $(JJ^\dagger)_{\bar{p}\bar{n}} \equiv J_{\bar{p}\bar{\alpha}} \delta_{\bar{\alpha}\bar{\alpha}} \bar{J}_{n\bar{\alpha}}$ is a $N \times N$ hermitian matrix of external sources. Note that J and J^\dagger are independent.

Step 2: The second ingredient consists in evaluating the integral over σ by a replica trick. Let X be a complex vector of components X_1, \dots, X_N . We want to compute an integral with normalized Gaussian measure of covariance C (denoted $d\mu_C(X)$) of some perturbation $V(X, \bar{X})$. We expand in V (of course all this is justified only provided that the final expression is absolutely convergent) to get

$$I = \int d\mu_C(X) e^{V(\bar{X}, X)} = \int d\mu_C(X) \sum_{n \geq 0} \frac{1}{n!} V(\bar{X}, X)^n . \quad (92)$$

The term of degree n can be rewritten as a Gaussian integral over n replicas $X^{(1)}, \dots, X^{(n)}$ with degenerate covariance between replicas $C_{a\bar{b}}^{(i,j)} = C_{a\bar{b}}$, hence

$$I = \sum_{n \geq 0} \frac{1}{n!} \int d\mu_{C_{a\bar{b}}^{(i,j)}}(X^{(1)}, \dots, X^{(n)}) \prod_{i=1}^n V(\bar{X}^{(i)}, X^{(i)}) . \quad (93)$$

We will regard each term in this expansion as a function of parameters w^{ij} , evaluated for all $w^{ij} = 1$, corresponding to a Gaussian measure with covariance $C_{a\bar{b}}^{(i,i)} = C_{a\bar{b}}$, $C_{a\bar{b}}^{(i,j)} = w^{ij} C_{a\bar{b}}$ $i \neq j$.

Step 3: The third ingredient is the universal Brydges-Kennedy-Abdesselam-Rivasseau forest formula [35]. Consider n sites labeled $1, 2, \dots, n$ and a function f depending on $\frac{n(n-1)}{2}$ link variables x^{ij} with $i \neq j$. Then

$$f(1, \dots, 1) = \sum_{\mathcal{F}_n} \int_0^1 \left(\prod_{(i,j) \in \mathcal{F}_n} du^{ij} \right) \left(\frac{\partial^{|\mathcal{E}(\mathcal{F}_n)|} f}{\prod_{(i,j) \in \mathcal{F}_n} \partial x^{ij}} \right) \Big|_{x^{kl} = w^{kl}(\mathcal{F}_n, u)},$$

$$w^{kl}(\mathcal{F}_n, u) = \inf_{(i,j) \in \mathcal{P}_{k \rightarrow l}(\mathcal{F}_n)} u^{ij}, \quad (94)$$

where \mathcal{F}_n runs over all the forests (i.e. graphs with no loop lines) with vertices labeled $1, 2, \dots, n$ built over the n sites, $|\mathcal{E}(\mathcal{F}_n)|$ denotes the number of lines in the forest \mathcal{F}_n , $\mathcal{P}_{k \rightarrow l}(\mathcal{F}_n)$ is the unique path in the forest \mathcal{F}_n joining the vertices k and l , and the infimum is set to zero if there is no such path (i.e. k and l belong to different trees in the forest).

We compute

$$\begin{aligned} & \frac{\partial^{|\mathcal{E}(\mathcal{F}_n)|}}{\prod_{(i,j) \in \mathcal{F}_n} \partial x^{ij}} \left[\int d\mu_{x^{ij} C_{a\bar{b}}}(X^{(1)}, \dots, X^{(n)}) \prod_{i=1}^n V(\bar{X}^{(i)}, X^{(i)}) \right] \\ &= \frac{\partial^{|\mathcal{E}(\mathcal{F}_n)|}}{\prod_{(i,j) \in \mathcal{F}_n} \partial x^{ij}} \left[e^{\sum_{ab;i,j} x^{ij} \frac{\delta}{\delta X_a^{(i)}} C_{a\bar{b}} \frac{\delta}{\delta \bar{X}_b^{(j)}}} \prod_{i=1}^n V(\bar{X}^{(i)}, X^{(i)}) \right]_{X^{(i)} = \bar{X}^{(i)} = 0} \\ &= \left[e^{\sum_{ab;i,j} x^{ij} \frac{\delta}{\delta X_a^{(i)}} C_{a\bar{b}} \frac{\delta}{\delta \bar{X}_b^{(j)}}} \left(\prod_{(i,j) \in \mathcal{F}_n} \left(\sum_{a\bar{b}} \frac{\delta}{\delta X_a^{(i)}} C_{a\bar{b}} \frac{\delta}{\delta \bar{X}_b^{(j)}} \right) \right) \prod_{i=1}^n V(\bar{X}^{(i)}, X^{(i)}) \right]_{X^{(i)} = \bar{X}^{(i)} = 0} \\ &= \int d\mu_{x^{ij} C_{a\bar{b}}}(X^{(1)}, \dots, X^{(n)}) \left(\prod_{(i,j) \in \mathcal{F}_n} \left(\sum_{a\bar{b}} \frac{\delta}{\delta X_a^{(i)}} C_{a\bar{b}} \frac{\delta}{\delta \bar{X}_b^{(j)}} \right) \right) \prod_{i=1}^n V(\bar{X}^{(i)}, X^{(i)}), \end{aligned} \quad (95)$$

hence the forest formula applied to the replicated integral yields (the repeated indices a and \bar{b} are summed)

$$I = \sum_{n \geq 0} \frac{1}{n!} \sum_{\mathcal{F}_n} \int_0^1 \left(\prod_{(i,j) \in \mathcal{F}_n} du^{ij} \right) \int d\mu_{w^{ij}(\mathcal{F}_n, u) C_{a\bar{b}}}(X^{(1)}, \dots, X^{(n)})$$

$$\left(\prod_{(i,j) \in \mathcal{F}_n} \frac{\delta}{\delta X_a^{(i)}} C_{a\bar{b}} \frac{\delta}{\delta \bar{X}_b^{(j)}} \right) \prod_{i=1}^n V(\bar{X}^{(i)}, X^{(i)}), \quad (96)$$

with $w^{ii}(\mathcal{F}_n, u) = 1$ and $w^{ij}(\mathcal{F}_n, u) = \inf_{(k,l) \in \mathcal{P}_{i \rightarrow j}(\mathcal{F}_n)} u^{kl}$. Thus in our case we get (taking into account that the measure over σ is $1 \otimes 1$)

$$Z(J, \bar{J}; \lambda, N) = \sum_{n \geq 0} \frac{1}{n!} \sum_{\mathcal{F}_n} \int_0^1 \left(\prod_{(i,j) \in \mathcal{F}_n} du^{ij} \right) \int d\mu_{w^{ij}(\mathcal{F}_n, u) 1 \otimes 1}(\sigma^{(1)}, \dots, \sigma^{(n)})$$

$$\left(\prod_{(i,j) \in \mathcal{F}_n} \frac{\delta}{\delta \sigma_{ab}^{(i)}} \frac{\delta}{\delta \sigma_{ba}^{(j)\dagger}} \right) \prod_{i=1}^n \left\{ N^{D-1} \text{tr} \ln [R(\sigma^{(i)})] + N^{D-1} \text{tr} [R(\sigma^{(i)}) J J^\dagger] \right\}. \quad (97)$$

Note that the Gaussian integral factors over the connected components of the forests (i.e. trees). The main advantage of eq. (97) is that it allows to compute $W(J, \bar{J}; \lambda, N)$ very easily: whenever a function is a sum over forests of contributions which factor over the trees, its logarithm is the sum over trees of the tree contribution, hence

$$W(J, \bar{J}; \lambda, N) = \sum_{n \geq 1} \frac{1}{n!} \sum_{\mathcal{T}_n} \int_0^1 \left(\prod_{(i,j) \in \mathcal{T}_n} du^{ij} \right) \int d\mu_{w^{ij}(\mathcal{T}_n, u) 1 \otimes 1}(\sigma^{(1)}, \dots, \sigma^{(n)})$$

$$N^{(D-1)n} \left(\prod_{(i,j) \in \mathcal{T}_n} \frac{\delta}{\delta \sigma_{ab}^{(i)}} \frac{\delta}{\delta \sigma_{ba}^{(j)\dagger}} \right) \prod_{i=1}^n \left\{ \text{tr} \ln [R(\sigma^{(i)})] + \text{tr} [R(\sigma^{(i)}) J J^\dagger] \right\}, \quad (98)$$

where \mathcal{T}_n runs over all trees with vertices labeled $1, 2, \dots, n$ and $w^{ii}(\mathcal{T}_n, u) = 1$ and $w^{ij}(\mathcal{T}_n, u) = \inf_{(k,l) \in \mathcal{P}_{i \rightarrow j}(\mathcal{T}_n)} u^{kl}$, with $\mathcal{P}_{i \rightarrow j}(\mathcal{T}_n)$ the path in the tree \mathcal{T}_n connecting i and j . Expanding the product, we get

$$\begin{aligned}
W(J, \bar{J}; \lambda, N) &= \\
&\sum_{n \geq 1} \frac{1}{n!} N^{(D-1)n} \sum_{\mathcal{T}_n} \int_0^1 \left(\prod_{(i,j) \in \mathcal{T}_n} du_{ij} \right) \int d\mu_{w^{ij}(\mathcal{T}_n, u) 1 \otimes 1}(\sigma^{(1)}, \dots, \sigma^{(n)}) \left(\prod_{(i,j) \in \mathcal{T}_n} \frac{\delta}{\delta \sigma_{ab}^{(i)}} \frac{\delta}{\delta \sigma_{ba}^{(j)\dagger}} \right) \\
&\sum_{k=0}^n \prod_{d=1}^k [JJ^\dagger]_{\bar{m}_d n_d} \sum_{i_1 < i_2 < \dots < i_k}^n R(\sigma^{(i_1)})_{n_1 \bar{m}_1} R(\sigma^{(i_2)})_{n_2 \bar{m}_2} \dots R(\sigma^{(i_k)})_{n_k \bar{m}_k} \prod_{\substack{i=1 \\ i \neq i_1, \dots, i_k}}^n \text{tr} \ln [R(\sigma^{(i)})] \\
&= \sum_{n \geq 1} \frac{1}{n!} N^{(D-1)n} \sum_{\mathcal{T}_n} \int_0^1 \left(\prod_{(i,j) \in \mathcal{T}_n} du_{ij} \right) \int d\mu_{w^{ij}(\mathcal{T}_n, u) 1 \otimes 1}(\sigma^{(1)}, \dots, \sigma^{(n)}) \left(\prod_{(i,j) \in \mathcal{T}_n} \frac{\delta}{\delta \sigma_{ab}^{(i)}} \frac{\delta}{\delta \sigma_{ba}^{(j)\dagger}} \right) \\
&\sum_{k=0}^n \frac{1}{k!} \prod_{d=1}^k [JJ^\dagger]_{\bar{m}_d n_d} \sum_{\substack{i_1, i_2, \dots, i_k=1 \\ i_d \neq i_{d'}}}^n R(\sigma^{(i_1)})_{n_1 \bar{m}_1} R(\sigma^{(i_2)})_{n_2 \bar{m}_2} \dots R(\sigma^{(i_k)})_{n_k \bar{m}_k} \prod_{\substack{i=1 \\ i \neq i_1, \dots, i_k}}^n \text{tr} \ln [R(\sigma^{(i)})] .
\end{aligned} \tag{99}$$

We represent every vertex of \mathcal{T}_n corresponding to a $\text{tr} \ln [R(\sigma^{(i)})]$ as a fat vertex, and every vertex corresponding to a term $R(\sigma^{(i_r)})_{n_r \bar{m}_r}$ as a flattened vertex (whose end points we label n_r and \bar{m}_r). We designate the flattened vertices as *external*. The vertices are labeled by the index i of the corresponding replicated field $\sigma^{(i)}$.

Each derivative with σ and σ^\dagger brings a resolvent (of the appropriate replica). The resolvents are contracted along the lines of the tree which (as the field σ has two indices) are double (ribbon) lines. The two indices of σ provide a well defined ordering of the tree lines touching each vertex generating, for each \mathcal{T}_n , a sum over the various unrooted labeled plane trees $\mathcal{T}_{n,k}^\circ$ with n vertices, out of which k (having labels i_1, \dots, i_k) are external, compatible with \mathcal{T}_n . In figure 13 we presented such a plane tree having two external vertices, 1 and 4 (we added a dashed strand for the flattened vertices).

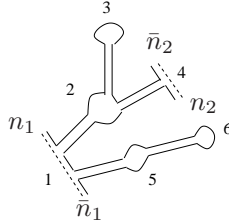


Figure 13: A labeled plane tree in the LVE.

Every tree \mathcal{T}_n with assigned degrees of the vertices d_1, \dots, d_n , has $d_{i_1}! \dots d_{i_k}! \prod_{i \neq i_k} (d_i - 1)!$ associated plane trees $\mathcal{T}_{n,k}^\circ$, corresponding to the permutations of all but one of the halflines touching each vertex (and a choice d_{i_r} of where to place the external insertion on the external vertices). As the number of combinatorial trees with assigned degrees d_1, \dots, d_n is $\frac{(n-2)!}{(d_1-1)! \dots (d_n-1)!}$ we get

$$\begin{aligned}
\sum_{\mathcal{T}_{n,k}^\circ} 1 &= \sum_{\substack{d_1, \dots, d_n=1 \\ \sum d_i=2n-2}}^n \frac{(n-2)!}{(d_1-1)! \dots (d_n-1)!} d_{i_1}! \dots d_{i_k}! \prod_{i \neq i_k} (d_i - 1)! \\
&= (n-2)! \sum_{\substack{d_1, \dots, d_n=1 \\ \sum d_i=2n-2}}^n d_{i_1} \dots d_{i_k} = (n-2)! \binom{2n+k-3}{n-2},
\end{aligned} \tag{100}$$

as the sums over d_i yield the coefficient of the term of degree x^{2n-2} in the expansion of

$$\left[x \left(\frac{1}{1-x} \right)' \right]^k \frac{x^{n-k}}{(1-x)^{n-k}} = \frac{x^n}{(1-x)^{n+k}} = x^n \sum_p \binom{n+k+p-1}{p} x^p. \tag{101}$$

The sides of the ribbon lines and of the fat vertices form open *strands*. The tree in figure 13 has two strands. To every plane tree with external vertices one canonically associates the ordered lists of vertices encountered when following the strands starting from their external index n_q of the vertex i_q (including the two end vertices of the strand). For the example of figure 13, this lists are 1, 2, 3, 2, 4 and 4, 2, 1, 5, 6, 5, 1. A moment's reflection reveals that the contribution of each plane tree is proportional to the ordered product of resolvents associated to the vertices along the strands: for the example in figure 13 it reads

$$\left[R(\sigma^{(1)})R(\sigma^{(2)})R(\sigma^{(3)})R(\sigma^{(2)})R(\sigma^{(4)}) \right]_{n_1 \bar{n}_2} \left[R(\sigma^{(4)})R(\sigma^{(2)})R(\sigma^{(1)})R(\sigma^{(5)})R(\sigma^{(6)})R(\sigma^{(5)})R(\sigma^{(1)}) \right]_{n_2 \bar{n}_1}. \quad (102)$$

We index the strands of $\mathcal{T}_{n,k}^\circ$ by their start and end vertices i_d and $i_{\xi(d)}$. We denote $\Xi[d \rightarrow \xi(d)]$ the ordered list of the vertices encountered following the strand (including the external vertices) starting from the index n_{i_d} of i_d and ending on $\bar{m}_{\xi(d)}$ on $i_{\xi(d)}$. As the strands are the boundary of the plane tree, ξ is a cyclic permutation (each cycle of ξ corresponds to a connected component of $\mathcal{T}_{n,k}^\circ$). Derivatives w.r.t σ and σ^\dagger bring factors $-\sqrt{\frac{\lambda}{N^{D-1}}}$ and $\sqrt{\frac{\lambda}{N^{D-1}}}$ respectively, and we finally obtain

$$\begin{aligned} W(J, \bar{J}; \lambda, N) &= N^{D-1} \sum_{n \geq 1} \frac{1}{n!} (-\lambda)^{n-1} \sum_{k=0}^n \frac{1}{k!} \sum_{\substack{i_1, i_2, \dots, i_k=1 \\ i_d \neq i_{d'}}}^n \sum_{\mathcal{T}_{n,k}^\circ} \int_0^1 \left(\prod_{(i,j) \in \mathcal{T}_n} du^{ij} \right) \\ &\quad \int d\mu_{w^{ij}(\mathcal{T}_n, u)1 \otimes 1}(\sigma^{(1)}, \dots, \sigma^{(n)}) \prod_{d=1}^k [JJ^\dagger]_{\bar{m}_d n_d} \left[\prod_{j \in \Xi[d \rightarrow \xi(d)]} R(\sigma^{(j)}) \right]_{n_d \bar{m}_{\xi(d)}} \\ &= N^{D-1} \sum_{n \geq 1} \frac{1}{n!} (-\lambda)^{n-1} \sum_{k=0}^n \frac{1}{k!} \sum_{\substack{i_1, i_2, \dots, i_k=1 \\ i_d \neq i_{d'}}}^n \sum_{\mathcal{T}_{n,k}^\circ} \int_0^1 \left(\prod_{(i,j) \in \mathcal{T}_n} du^{ij} \right) \\ &\quad \int d\mu_{w^{ij}(\mathcal{T}_n, u)1 \otimes 1}(\sigma^{(1)}, \dots, \sigma^{(n)}) \operatorname{tr} \left[\prod_{d=1}^k \left(JJ^\dagger \prod_{j \in \Xi[d \rightarrow \xi(d)]} R(\sigma^{(j)}) \right) \right]. \end{aligned} \quad (103)$$

where \mathcal{T}_n is the unique combinatorial tree to which the plane tree $\mathcal{T}_{n,k}^\circ$ reduces. The product over d is the ordered product of resolvents and external sources encountered when going around the tree. We chose as start point of this product the vertex i_1 but, as the trace is cyclic, one can chose any other vertex i_2, i_3 and so on as the start vertex. We can now prove our first result concerning the LVE expansion.

Theorem 5. *The series in eq. (103) is absolutely convergent for $0 \leq \lambda < 3^{-2}$ and $\|JJ^\dagger\| < 3^{-1}$ (hence uniformly in N).*

Proof: We bound $\operatorname{tr}(\prod A_i) \leq N \prod \|A_i\|$, and take into account $\|R(\sigma)\| \leq 1$ for $\lambda \geq 0$. The Gaussian integrals are normalized, and the integrals over the parameters u are bounded by 1, thus

$$\begin{aligned} |W(J, \bar{J}; \lambda, N)| &\leq N^D \sum_{n \geq 1} \frac{1}{n!} \lambda^{n-1} \sum_{k=0}^n \frac{1}{k!} \|JJ^\dagger\|^k \sum_{\substack{i_1, i_2, \dots, i_k=1 \\ i_d \neq i_{d'}}}^n \sum_{\mathcal{T}_{n,k}^\circ} 1 \\ &= \sum_{n \geq 1} \frac{1}{n!} \lambda^{n-1} \sum_{k=0}^n \frac{1}{k!} \|JJ^\dagger\|^k \frac{n!}{(n-k)!} (n-2)! \binom{2n+k-3}{n-2} \\ &= N^D \sum_{n \geq 1} \lambda^{n-1} \sum_{k=0}^n \|JJ^\dagger\|^k \frac{(2n+k-3)!}{k!(n-k)!(n+k-1)!}, \end{aligned} \quad (104)$$

and using $\frac{(2n+k-3)!}{k!(n-k)!(n+k-1)!} < 3^{2n+k-1}$ we get

$$|W(J, \bar{J}; \lambda, N)| \leq \frac{1}{3\lambda} N^D \sum_{n=1} (3^2 \lambda)^n \sum_{k=0}^n (3 \|JJ^\dagger\|)^k, \quad (105)$$

which converges for $0 \leq \lambda < 3^{-2}$ and $\|JJ^\dagger\| < 3^{-1}$. □

Equation (103) yields the non perturbative definition of the generating function of the cumulants of our probability measure (in the interval $\lambda \in [0, 3^{-2})$ of the real axis). In order to show that $W(J, \bar{J}; \lambda, N)$ thus defined is the Borel sum of connected Feynman graphs (i.e. it is the Borel sum of its Taylor expansion around $\lambda = 0$) we must show that (103) is a Borel summable function in λ uniformly in N .

Theorem 6 (Nevanlinna-Sokal). *A function $f(\lambda, N)$ is said to be Borel summable in λ uniformly in N if*

- $f(\lambda, N)$ is analytic in a disk $\Re \lambda^{-1} > R^{-1}$ with R independent of N
- $f(\lambda, N)$ admits a Taylor expansion at the origin

$$f(\lambda, N) = \sum_{k=0}^{r-1} f_{N,k} \lambda^k + R_{N,r}(\lambda), \quad |R_{N,r}(\lambda)| \leq K \sigma^r r! |\lambda|^r \quad (106)$$

for some constants K and σ independent of N .

If $f(\lambda, N)$ is Borel summable in λ uniformly in N then $B(t, N) = \sum_{k=0}^{\infty} \frac{1}{k!} f_{N,k} t^k$ is an analytic function for $|t| < \sigma^{-1}$ which admits an analytic continuation in the strip $\{z \mid |\Im z| < \sigma^{-1}\}$ such that $|B(t, N)| < B e^{t/R}$ for some constant B independent of N and $f(\lambda, N)$ is represented by the absolutely convergent integral

$$f(\lambda, N) = \frac{1}{\lambda} \int_0^{\infty} dt B(t, N) e^{-\frac{t}{\lambda}} \quad (107)$$

That is the Taylor expansion of $f(\lambda, N)$ at the origin is Borel summable, and $f(\lambda, N)$ is its Borel sum.

Theorem 7. *The function $N^{-D}W(J, \bar{J}; \lambda, N)$ with $W(J, \bar{J}; \lambda, N)$ defined in eq. (103) is Borel summable in λ uniformly in N for $\|JJ^\dagger\|$ small enough.*

Proof: Consider a complex $\lambda = |\lambda| e^{i\varphi}$. Following step by step the proof of theorem 5 and using $\|1 + \rho e^{i\alpha}\| > |\sin \alpha| \Rightarrow \|R(\sigma)\| < \frac{1}{|\cos(\frac{\varphi}{2})|}$ it follows that (103) is convergent for $|\lambda| < 3^{-2} |\cos \frac{\varphi}{2}|^2 = \frac{\cos \varphi + 1}{18}$, hence it certainly converges in the Borel disk $18 < \Re \lambda^{-1}$.

To compute the remainder $R_{N,r}(\lambda)$ we separate $N^{-D}W(J, \bar{J}; \lambda, N)$ into two terms: the terms with $n < r + 1$ and the ones with $n \geq r + 1$. The terms with $n \geq r + 1$ are all in the reminder and admit the bound

$$\begin{aligned} & \left| N^{-1} \sum_{n=r+1}^{\infty} \frac{1}{n!} (-\lambda)^{n-1} \sum_{k=0}^n \frac{1}{k!} \sum_{\substack{i_1, i_2, \dots, i_k=1 \\ i_d \neq i_{d'}}}^n \sum_{\mathcal{T}_{n,k}^\circ} \int_0^1 \left(\prod_{(i,j) \in \mathcal{T}_n} du^{ij} \right) \right. \\ & \left. \int d\mu_{w^{ij}(\mathcal{T}_n, u)1 \otimes 1}(\sigma^{(1)}, \dots, \sigma^{(n)}) \operatorname{tr} \left[\prod_{d=1}^k \left(JJ^\dagger \prod_{j \in \Xi[\xi^{d-1}(1) \rightarrow \xi^d(1)]} R(\sigma^{(j)}) \right) \right] \right| \\ & \leq \sum_{n=r+1}^{\infty} |\lambda|^{n-1} \sum_{k=0}^n \|JJ^\dagger\|^k \frac{(2n+k-3)!}{k!(n-k)!(n+k-1)!} \leq \sum_{n=r+1}^{\infty} |\lambda|^{n-1} \sum_{k=0}^n \|JJ^\dagger\|^k 3^{2n+k-1} \leq |\lambda|^r K^r. \end{aligned} \quad (108)$$

with K some constant and both $\|JJ^\dagger\|$ and λ small enough, which is certainly bounded by $K^r r! |\lambda|^r$. It remains to find a good bound for the contribution to the reminder of the terms with $n < r + 1$,

$$N^{-1} \sum_{n=1}^r \frac{1}{n!} (-\lambda)^{n-1} \sum_{k=0}^n \frac{1}{k!} \sum_{\substack{i_1, i_2, \dots, i_k=1 \\ i_d \neq i_{d'}}}^n \sum_{\mathcal{T}_{n,k}^\circ} \int_0^1 \left(\prod_{(i,j) \in \mathcal{T}_n} du^{ij} \right) \quad (109)$$

$$\int d\mu_{w^{ij}(\mathcal{T}_n, u)1 \otimes 1}(\sigma^{(1)}, \dots, \sigma^{(n)}) \operatorname{tr} \left[\prod_{d=1}^k \left(JJ^\dagger \prod_{j \in \Xi[\xi^{d-1}(1) \rightarrow \xi^d(1)]} R(\sigma^{(j)}) \right) \right]. \quad (110)$$

For each plane tree, we use a Taylor expansion in with integral reminder of the product of resolvents up to some order to be chosen later

$$f(\sqrt{\lambda}) = \sum_{q=0}^{s-1} \frac{1}{q!} \lambda^{\frac{q}{2}} f^{(q)}(0) + \frac{1}{(s-1)!} \int_0^1 (1-t)^{s-1} \frac{d^s f}{dt^s}(t\sqrt{\lambda}) dt. \quad (111)$$

The first terms yield some series in λ , as the Gaussian integral is non zero only for an even number of insertions. For every resolvent appearing in the product we have, taking into account that $\sigma - \sigma^\dagger$ commutes with $R(\sigma)$,

$$\partial_t \left[\frac{1}{1 + t\sqrt{\frac{\lambda}{N^{D-1}}}(\sigma - \sigma^\dagger)} \right]_{ab} = -\sqrt{\frac{\lambda}{N^{D-1}}} [R(t\sigma)(\sigma - \sigma^\dagger)R(t\sigma)]_{ab} = \left(\sigma_{np} \frac{\delta}{\delta \sigma_{np}} + \sigma_{np}^\dagger \frac{\delta}{\delta \sigma_{np}^\dagger} \right) R(t\sigma)_{ab}, \quad (112)$$

where the (repeated) indices n and p are summed. Taking into account the copies we get,

$$\begin{aligned} & \partial_t \left(\text{tr} \left[\prod_{d=1}^k \left(J J^\dagger \prod_{j \in \Xi[\xi^{k-1}(1) \rightarrow \xi^k(1)]} R(t\sigma^{(j)}) \right) \right] \right) \\ &= \sum_i \left(\sigma_{np}^{(i)} \frac{\delta}{\delta \sigma_{np}^{(i)}} + \sigma_{np}^{(i)\dagger} \frac{\delta}{\delta \sigma_{np}^{(i)\dagger}} \right) \text{tr} \left[\prod_{d=1}^k \left(J J^\dagger \prod_{j \in \Xi[\xi^{d-1}(1) \rightarrow \xi^d(1)]} R(t\sigma^{(j)}) \right) \right], \end{aligned} \quad (113)$$

Integrating by parts the Gaussian integral we get

$$\begin{aligned} & \partial_t \left[\int_0^1 \left(\prod_{(i,j) \in \mathcal{T}_n} du_{ij} \right) \int d\mu_{w^{ij}(\mathcal{T}_n, u)1 \otimes 1}(\sigma^{(1)}, \dots, \sigma^{(n)}) \text{tr} \left[\prod_{d \in C(\xi)} \vec{\eta} \eta^\dagger \left(\prod_{j \in \Xi[d \rightarrow \xi(d)]} R(\sigma^{(j)}) \right) \right] \right] \\ &= \int_0^1 \left(\prod_{(i,j) \in \mathcal{T}_n} du_{ij} \right) \int d\mu_{w^{ij}(\mathcal{T}_n, u)1 \otimes 1}(\sigma^{(1)}, \dots, \sigma^{(n)}) \left(2 \sum_{i,j} w^{ij} \frac{\delta}{\delta \sigma_{np}^{(i)}} \frac{\delta}{\delta \sigma_{np}^{(j)\dagger}} \right) \\ & \quad \text{tr} \left[\prod_{d=1}^k \left(J J^\dagger \prod_{j \in \Xi[\xi^{d-1}(1) \rightarrow \xi^d(1)]} R(t\sigma^{(j)}) \right) \right]. \end{aligned} \quad (114)$$

It follows that the derivative of order s is

$$\begin{aligned} & \frac{\partial^s}{\partial t^s} \left[\int_0^1 \left(\prod_{(i,j) \in \mathcal{T}_n} du_{ij} \right) \int d\mu_{w^{ij}(\mathcal{T}_n, u)1 \otimes 1}(\sigma^{(1)}, \dots, \sigma^{(n)}) \text{tr} \left[\prod_{d=1}^k \left(J J^\dagger \prod_{j \in \Xi[\xi^{d-1}(1) \rightarrow \xi^d(1)]} R(t\sigma^{(j)}) \right) \right] \right] \\ &= \int_0^1 \left(\prod_{(i,j) \in \mathcal{T}_n} du_{ij} \right) \int d\mu_{w^{ij}(\mathcal{T}_n, u)1 \otimes 1}(\sigma^{(1)}, \dots, \sigma^{(n)}) \left(2 \sum_{i,j} w^{ij} \frac{\delta}{\delta \sigma_{np}^{(i)}} \frac{\delta}{\delta \sigma_{np}^{(j)\dagger}} \right)^s \\ & \quad \text{tr} \left[\prod_{d=1}^k \left(J J^\dagger \prod_{j \in \Xi[\xi^{d-1}(1) \rightarrow \xi^d(1)]} R(t\sigma^{(j)}) \right) \right]. \end{aligned} \quad (115)$$

When computing explicitly the derivative operators acting on the trace one generates ribbon loop lines decorating the plane tree $\mathcal{T}_{n,k}$. The traces recombine to reconstitute the product of $R(\sigma)$ and $J J^\dagger$ on each face of this graph. An example is presented in figure 14 consisting in the tree of figure 13 decorated by two loop lines. Its contribution is

$$\begin{aligned} & \text{tr} [R(t\sigma^{(2)})R(t\sigma^{(3)})] \quad \text{tr} [R(t\sigma^{(5)})R(t\sigma^{(6)})] \\ & \text{tr} [J J^\dagger R(t\sigma^{(1)})R(t\sigma^{(2)})R(t\sigma^{(3)})R(t\sigma^{(2)})R(t\sigma^{(4)})J J^\dagger \\ & \quad R(t\sigma^{(4)})R(t\sigma^{(2)})R(t\sigma^{(1)})R(t\sigma^{(5)})R(t\sigma^{(6)})R(t\sigma^{(5)})R(t\sigma^{(1)})] \end{aligned} \quad (116)$$

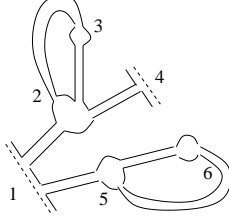


Figure 14: A plane tree decorated by loop lines.

Bounding again the resolvents by 1, and the traces by N times the norm, each such term is bounded by

$$2^s \left(-\frac{\lambda}{N^{D-1}} \right)^s \|JJ^\dagger\|^k N^{1+s}, \quad (117)$$

as the number of faces of the ribbon graph obtained from the plane tree $\mathcal{T}_{n,k}^\circ$ by adding the s loop lines is at most $1 + s$.

The initial tree has $2(n-1) + k$ resolvents. Every derivative brings a new resolvent, hence the number of contractions (the number of ways one can connect the loop lines on the tree) is

$$[2(n-1) + k][2(n-1) + k + 1] \dots [2(n-1) + k + 2s - 1] = \frac{[2(n-1) + k + 2s - 1]!}{[2(n-1) + k - 1]!}. \quad (118)$$

Choosing $s = r - (n-1)$, and taking into account that $w^{ij} < 1$, the Gaussian integrals are normalized and the integral over dt is bounded by 1, the remainder term is bounded by

$$\begin{aligned} & N^{-1} \sum_{n=1}^r \frac{1}{n!} |\lambda|^{n-1} \sum_{k=0}^n \frac{1}{k!} \frac{n!}{(n-k)!} (n-2)! \binom{2n+k-3}{n-2} \\ & \frac{[2(n-1) + k - 1 + 2r - 2(n-1)]!}{(r-n)! [2(n-1) + k - 1]!} 2^{r-n+1} \left(\frac{|\lambda|}{N^{D-1}} \right)^{r-n+1} \|JJ^\dagger\|^k N^{r-n+2} \\ & \leq |\lambda|^r 2^r \sum_{n=1}^r \sum_{k=0}^n \|JJ^\dagger\|^k \frac{1}{k! (n-k)!} \frac{(2n+k-3)!}{(n+k-1)!} \frac{(2r+k-1)!}{(r-n)! [2(n-1) + k - 1]!}. \end{aligned} \quad (119)$$

We have $\frac{1}{k!(n-k)!} \frac{(2n+k-3)!}{(n+k-1)!} < 3^{2n+k-1} < 3^{3r}$ and $\frac{(2r+k-1)!}{(r-n)! [2(n-1) + k - 1]!} < 3^{2r+k-1} (r-n+2)! < 3^{3r} (r+1)!$. Moreover $\sum_{n=1}^r \sum_{k=0}^n 1 < (r+1)^2$ thus for $\|JJ^\dagger\| < 1$ we get a bound on the contribution of the first terms to the reminder

$$(23^3 3^3)^r |\lambda|^r (r+1)^3 r! < (23^3 3^3 e^2)^r r! |\lambda|^r. \quad (120)$$

Note that although the bound we have established might not appear tight, in fact it is: the $r!$ growth of the reminder is not an artifact, but it is generated by the proliferation of the Wick contractions in a graph with loop lines. □

We have thus far given a non perturbative definition for the cumulants, eq. (103), and showed that it is the Borel sum of its perturbative expansion in Feynman graphs. We now prove that the measure thus defined is properly uniformly bounded.

Theorem 8. *The perturbed Gaussian measure in eq. (84) is trace invariant and properly uniformly bounded.*

Proof: We use the invariance under unitary transformations. We add a fictitious integral over the unitary group $U(N)$, i.e. we write

$$W(J, \bar{J}; \lambda, N) = \int_{U(N)} [dU] W(J, \bar{J}; \lambda, N), \quad (121)$$

which of course holds as nothing depends of U on the right hand side. Now, for all fixed U , we perform the change of variables of Jacobian 1, $\sigma^{(i)} \rightarrow U^\dagger \sigma^{(i)} U$. in eq. (103). The Gaussian measure is invariant under this change of variables, hence

$$W(J, \bar{J}; \lambda, N) = N^{D-1} \sum_{n \geq 1} \frac{1}{n!} (-\lambda)^{n-1} \sum_{k=0}^n \frac{1}{k!} \sum_{\substack{i_1, i_2, \dots, i_k=1 \\ i_d \neq i_{d'}}}^n \sum_{\mathcal{T}_{n,k}^\circ} \int_0^1 \left(\prod_{(i,j) \in \mathcal{T}_n} du^{ij} \right) \\ \int d\mu_{w^{ij}(\mathcal{T}_{n,u})1 \otimes 1}(\sigma^{(1)}, \dots, \sigma^{(n)}) \\ \int [dU] \prod_{d=1}^k [U J J^\dagger U^\dagger]_{l_{\xi^{d-1}(1)} j_{\xi^{d-1}(1)}} \left[\prod_{j \in \Xi[\xi^{d-1}(1) \rightarrow \xi^d(1)]} R(\sigma^{(j)}) \right]_{j_{\xi^{d-1}(1)} l_{\xi^d(1)}}. \quad (122)$$

Rearranging the terms in the product over d we have

$$W(J, \bar{J}; \lambda, N) = N^{D-1} \sum_{n \geq 1} \frac{1}{n!} (-\lambda)^{n-1} \sum_{k=0}^n \frac{1}{k!} \sum_{\substack{i_1, i_2, \dots, i_k=1 \\ i_d \neq i_{d'}}}^n \sum_{\mathcal{T}_{n,k}^\circ} \int_0^1 \left(\prod_{(i,j) \in \mathcal{T}_n} du^{ij} \right) \\ \int d\mu_{w^{ij}(\mathcal{T}_{n,u})1 \otimes 1}(\sigma^{(1)}, \dots, \sigma^{(n)}) \\ \int [dU] \prod_{d=1}^k [U J J^\dagger U^\dagger]_{l_{djd}} \left[\prod_{j \in \Xi[d \rightarrow \xi(d)]} R(\sigma^{(j)}) \right]_{j_d l_{\xi(d)}}. \quad (123)$$

The integral over the unitary group of a product of matrix elements is (see [23])

$$\int_{U(N)} [dU] \prod_{d=1}^k U_{l_d \alpha_d} U_{\beta_d j_d}^\dagger = \sum_{\sigma, \tau} \text{Wg}(N, \sigma \tau^{-1}) \prod_{d=1}^k \delta_{l_d j_{\sigma(d)}} \delta_{\alpha_d \beta_{\tau(d)}}, \quad (124)$$

where σ and τ run over the permutations of k elements and $\text{Wg}(N, \sigma)$ is Weingarten's function [23], thus

$$W(J, \bar{J}; \lambda, N) = N^{D-1} \sum_{n \geq 1} \frac{1}{n!} (-\lambda)^{n-1} \sum_{k=0}^n \frac{1}{k!} \sum_{\substack{i_1, i_2, \dots, i_k=1 \\ i_d \neq i_{d'}}}^n \sum_{\mathcal{T}_{n,k}^\circ} \int_0^1 \left(\prod_{(i,j) \in \mathcal{T}_n} du^{ij} \right) \\ \int d\mu_{w^{ij}(\mathcal{T}_{n,u})1 \otimes 1}(\sigma^{(1)}, \dots, \sigma^{(n)}) \left(\prod_{d=1}^k [J J^\dagger]_{\alpha_d \beta_d} \left[\prod_{j \in \Xi[d \rightarrow \xi(d)]} R(\sigma^{(j)}) \right]_{j_d l_{\xi(d)}} \right) \\ \sum_{\sigma, \tau} \text{Wg}(N, \sigma \tau^{-1}) \prod_{d=1}^k \delta_{l_d j_{\sigma(d)}} \delta_{\alpha_d \beta_{\tau(d)}}.$$

The external sources group into a product of traces. Following the indices we see that $\beta_d \rightarrow \alpha_d \rightarrow \beta_{\tau(d)} \dots$ thus each trace of a product of insertions reproduces a cycle in the permutation τ . Denoting these cycles $C_r(\tau)$, and their length $|C_r(\tau)|$ (hence τ writes $\tau = C_1(\tau) \dots C_{c(\tau)}(\tau)$) we get

$$\prod_{d=1}^k [J J^\dagger]_{\alpha_d \beta_d} \delta_{\alpha_d \beta_{\tau(d)}} = \prod_{r=1}^{c(\tau)} \text{tr}[(J J^\dagger)^{|C_r(\tau)|}]. \quad (125)$$

Similarly, the indices j, l follow the cycles of the permutation σ as $j_d \rightarrow l_{\xi(d)} \rightarrow j_{\sigma\xi(d)} \dots$, thus the generating function of the cumulants is

$$W(J, \bar{J}; \lambda, N) = N^{D-1} \sum_{n \geq 1} \frac{1}{n!} (-\lambda)^{n-1} \sum_{k=0}^n \frac{1}{k!} \sum_{\substack{i_1, i_2, \dots, i_k=1 \\ i_d \neq i_{d'}}}^n \sum_{\mathcal{T}_{n,k}^\circ} \int_0^1 \left(\prod_{(i,j) \in \mathcal{T}_n} du^{ij} \right) \\ \int d\mu_{w^{ij}(\mathcal{T}_{n,u})1 \otimes 1}(\sigma^{(1)}, \dots, \sigma^{(n)}) \sum_{\sigma, \tau} \text{Wg}(N, \sigma \tau^{-1}) \prod_{r=1}^{c(\tau)} \text{tr}[(J J^\dagger)^{|C_r(\tau)|}]$$

$$\prod_{h=1}^{c(\sigma\xi)} \text{tr} \left[\prod_{d=1}^{|C_h(\sigma\xi)|} \left(\prod_{j \in \Xi[(\sigma\xi)^{d-1}(q) \rightarrow \xi(\sigma\xi)^{d-1}(q)]} R(\sigma^{(j)}) \right) \right]. \quad (126)$$

with q any element in the cycle $C_h(\sigma\tau)$. The cumulants are defined according to eq. (86) as the partial derivatives of $W(J, \bar{J}; \lambda, N)$. It follows that the distribution is trace invariant, as the non trivial cumulants at order $2k$ write as sums over graphs \mathcal{B} , whose connected components are the cycles over the external insertions JJ^\dagger . To each graph one has several possible τ associated permutations. The number of cycles of τ is the number of connected components of the graph \mathcal{B} , $\rho(\mathcal{B}) = c(\tau)$. We now show that the cumulants are properly uniformly bounded. The Weingarten function respects

$$\lim_{N \rightarrow \infty} N^{2k-c(\sigma)} \text{Wg}(N, \sigma) = \prod_{s=1}^{c(\sigma)} (-1)^{|C_s(\sigma)|} \frac{1}{|C_s(\sigma)| + 1} \binom{2|C_s(\sigma)|}{|C_s(\sigma)|}, \quad (127)$$

bounding the traces of products of resolvent by N , and taking into account that the Gaussian integrals are normalized, and the integrals over the parameters u are bounded by 1, we get a bound for each graph \mathcal{B} contributing to a cumulant of order $2k$ (using $K(\mathcal{B})$ as a dustbin notation for a constant independent of N , but depending on \mathcal{B})

$$|\mathfrak{K}(\mathcal{B}, \mu_N)| \leq K(\mathcal{B}) N^{-2k(D-1)} N^{D-1} \sum_{n=k}^{\infty} \frac{1}{n!} |\lambda|^n \frac{1}{k!} \sum_{\substack{i_1, i_2, \dots, i_k=1 \\ i_d \neq i_{d'}}}^n \sum_{\substack{\mathcal{T}_{n,k}^\circ \\ \sigma, \tau \in \mathfrak{S}(k)}} N^{-2k+c(\sigma\tau^{-1})+c(\sigma\xi)} \quad (128)$$

By lemma 8, $c(\xi) + c(\sigma\xi) + c(\tau) + c(\sigma\tau^{-1}) \leq 2 + 2k$ and taking into account that ξ is a cyclic permutation, $c(\xi) = 1$, and that the sums over τ and σ do not depend on n , we get a bound

$$K(\mathcal{B}) N^{D-2k(D-1)-\rho(\mathcal{B})} \sum_{n=k}^{\infty} |\lambda|^n \frac{(2n+k-3)!}{(n-k)!(n+k-1)!}, \quad (129)$$

for some constant $K(\mathcal{B})$, and the sum over n converges by the usual bounds. \square

B Other scalings of the cumulants

A natural question is to what extent the results presented in this paper can be generalized for different scalings of the cumulants. As already mentioned the scaling N^{D-1} of the Gaussian is the unique scaling which leads to convergent expectations for **all** invariants, not only for subclasses of invariants.

An interesting question is what happens if one allows the scaling of the cumulants to depend on finer details of the associated graphs. Of course if this extra scaling suppresses some of the cumulants the results hold. The interesting question is how much these scaling can be boosted, while still having a large N limit (universal or not). One particular scaling one can consider is to boost each invariant by a factor $N^{\Omega(\mathcal{B})}$

$$\kappa_{2k}[\mathbb{T}_{\vec{n}_1}, \bar{\mathbb{T}}_{\vec{n}_1} \dots \bar{\mathbb{T}}_{\vec{n}_k}] = \sum_{\substack{\mathcal{B} = \bigcup_{\rho=1}^{\rho(\mathcal{B})} \mathcal{B}_\rho \\ k(\mathcal{B})=k}} N^{-2(D-1)k(\mathcal{B})+D-\rho(\mathcal{B})+\Omega(\mathcal{B})} K(\mathcal{B}, N) \prod_{\rho=1}^{\rho(\mathcal{B})} \delta_{n\vec{n}}^{\mathcal{B}_\rho}, \quad (130)$$

with $\Omega(\mathcal{B})$ its convergence order (note that the convergence order, like the degree, factors over the connected components of the graph $\Omega(\mathcal{B}) = \sum_{\rho} \Omega(\mathcal{B}_\rho)$). The expectation of an observable writes again as a sum over doubled graphs \mathcal{G} ,

$$N^{-1+\Omega(\mathcal{B})} \mu \left(\text{Tr}_{\mathcal{B}}(\mathbb{T}, \bar{\mathbb{T}}) \right) = \sum_{\mathcal{G} \supset \mathcal{B}} \prod_{\alpha} K(\mathcal{B}(\alpha)) N^{-\frac{2}{(D-1)\Gamma} \omega(\mathcal{G}) + \frac{2}{(D-1)\Gamma} \min_{\mathcal{G}' \setminus \mathcal{E}^0 = \mathcal{B}} \omega(\mathcal{G}') + \sum_{\alpha, \rho} \min_{\mathcal{G}_\rho(\alpha) \setminus \mathcal{E}^0 = \mathcal{B}_\rho(\alpha)} \omega(\mathcal{G}_\rho(\alpha)) - D \sum_{\alpha} \left(\rho(\mathcal{B}(\alpha)) - 1 \right)}, \quad (131)$$

where we have expressed the convergence orders $\Omega(\mathcal{B})$ and $\Omega(\mathcal{B}_\rho(\alpha))$ as

$$\begin{aligned}\Omega(\mathcal{B}) &= \frac{2}{(D-1)!} \min_{\mathcal{G}' \setminus \mathcal{E}^0 = \mathcal{B}} \omega(\mathcal{G}') - \frac{2}{(D-2)!} \omega(\mathcal{B}) \\ \Omega(\mathcal{B}_\rho(\alpha)) &= \frac{2}{(D-1)!} \min_{\mathcal{G}_\rho(\alpha) \setminus \mathcal{E}^0 = \mathcal{B}_\rho(\alpha)} \omega(\mathcal{G}_\rho(\alpha)) - \frac{2}{(D-2)!} \omega(\mathcal{B}_\rho(\alpha)) ,\end{aligned}\quad (132)$$

with \mathcal{G}' and $\mathcal{G}_\rho(\alpha)$ covering graphs of \mathcal{B} and $\mathcal{B}_\rho(\alpha)$. Again the contribution of \mathcal{G} is dominant only if all the cumulants have a unique connected component $\rho(\mathcal{B}(\alpha)) = 1$, that is $\mathcal{B}(\alpha) \equiv \mathcal{B}_1(\alpha)$, which will be the case we consider from now on. Let us denote the total scaling with N in eq. (131)

$$\begin{aligned}\Lambda(\mathcal{G}) &\equiv -\frac{2}{(D-1)!} \omega(\mathcal{G}) + \frac{2}{(D-1)!} \min_{\mathcal{G}' \setminus \mathcal{E}^0 = \mathcal{B}} \omega(\mathcal{G}') + \sum_{\alpha} \min_{\mathcal{G}(\alpha) \setminus \mathcal{E}^0 = \mathcal{B}(\alpha)} \omega(\mathcal{G}(\alpha)) \\ &= \sum_i F^{0i}(\mathcal{G}) + D|\alpha| - \sup_{\mathcal{G}' \setminus \mathcal{E}^0 = \mathcal{B}} \sum_i F^{0i}(\mathcal{G}') - \sum_{\alpha} \sup_{\mathcal{G}(\alpha) \setminus \mathcal{E}^0 = \mathcal{B}(\alpha)} \sum_i F^{0i}(\mathcal{G}(\alpha)) .\end{aligned}\quad (133)$$

If $\mathcal{B}(\alpha)$ are all dipoles $\mathcal{B}^{(2)}$, corresponding to $\mathcal{G} = \mathcal{G}^{\min} \cup_{\mathcal{E}^0(\mathcal{G}^{\min})} \mathcal{B}^{(2)}$ for the minimal covering graphs \mathcal{G}^{\min} of \mathcal{B} , we obtain $\Lambda(\mathcal{G}^{\min} \cup_{\mathcal{E}^0(\mathcal{G}^{\min})}) = 0$, as all $\mathcal{G}(\alpha)$ are of degree 0. This reproduces the usual Gaussian evaluation. For the other doubled graphs \mathcal{G} there are three possible scenarios

- for all $\mathcal{G} \supset \mathcal{B}$, $\mathcal{G} \neq \mathcal{G}^{\min} \cup_{\mathcal{E}^0(\mathcal{G}^{\min})} \mathcal{B}^{(2)}$, $\Lambda(\mathcal{G}) < 0$. In this case the model admits a *Gaussian large N limit*.
- for all $\mathcal{G} \supset \mathcal{B}$, $\mathcal{G} \neq \mathcal{G}^{\min} \cup_{\mathcal{E}^0(\mathcal{G}^{\min})} \mathcal{B}^{(2)}$, $\Lambda(\mathcal{G}) \leq 0$, and there exists $\mathcal{G} \neq \mathcal{G}^{\min} \cup_{\mathcal{E}^0(\mathcal{G}^{\min})} \mathcal{B}^{(2)}$ with $\Lambda(\mathcal{G}) = 0$. In this case the model admits a *large N limit which is not Gaussian*.
- there exists $\mathcal{G} \supset \mathcal{B}$ with $\Lambda(\mathcal{G}) > 0$. In this case the model *does not admit a large N limit*.

The example of figure 15 shows that we are not in the first case. It consists in an observable and a

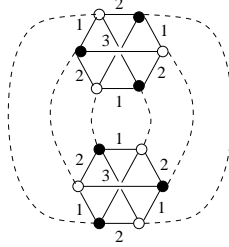


Figure 15: A doubled graph scaling like the Gaussian contribution.

cumulant in $D = 3$. The minimal graphs for both the observable and the cumulant have 6 faces of colors $0i$ (hence degree $\omega(\mathcal{G}(\alpha)^{\min}) = \omega(\mathcal{G}^{\min}) = 3$). The doubled graph has 9 faces of colors $0i$ (that is degree $\omega(\mathcal{G}) = 6$) thus $\Lambda(\mathcal{G}) = 0$.

It is for now an open question to discern in which of the remaining two cases we are. One can show using a Cauchy-Schwarz inequality that for any observable with $2k(\mathcal{B})$ vertices $\sum_i F^{0i}(\mathcal{G}) \leq Dk(\mathcal{B})$ for all doubled graphs $\mathcal{G} \supset \mathcal{B}$, and that the bound is saturated (by the case in which one has only one cumulant whose associated graph is the mirror image of the observable \mathcal{B} , as it is the case in figure 15). It follows that the model admits a large N limit if and only if, for all connected graphs \mathcal{B} , one has

$$\max_{\mathcal{G} \setminus \mathcal{E}^0 = \mathcal{B}} \sum_i F^{0i}(\mathcal{G}) \geq \frac{Dk(\mathcal{B}) + D}{2} .\quad (134)$$

While we have not been able to find any example in which this inequality is violated, we have not been able to prove it either.

Should this inequality hold, one would get a non Gaussian large N limit, which of course would be very interesting. Note however that if the models admits a large N limit with these scalings, then the leading order is rather non trivial. The example in figure 15 shows that at leading order one gets contributions from graphs which do not correspond to manifolds. Also, it is not clear (and it does seem unlikely) that the leading order graphs form a summable family like the planar or the melonic graphs.

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